Mirror Principle II

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Dedicated to Professor Michael Atiyah.

Abstract. We generalize our theorems in *Mirror Principle I* to a class of balloon manifolds. Many of the results are proved for convex projective manifolds. In a subsequent paper, Mirror Principle III, we will extend the results to projective manifolds without the convexity assumption.

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1. Introduction

For the long history of mirror symmetry, consult [17]. For a brief description of more recent development, see the introduction in [37][38]. The present paper is a sequel to *Mirror Principle I* [37]. Here, we generalize all the results there to a class of T-manifolds which we call balloon manifolds. These results were announced in [38].

Let X be a projective n-fold, and $d \in H^2(X, \mathbf{Z})$. Let $M_{0,k}(d, X)$ denote the moduli space of k-pointed, genus 0, degree d, stable maps $(C, f, x_1, .., x_k)$ on X [32]. Note that our notation is without the bar. By the work of [34](cf. [7][19]), each nonempty $M_{0,k}(d, X)$ admits a cycle class $LT_{0,k}(d, X)$ in the Chow group of degree $dim\ X + \langle c_1(X), d\rangle + n - 3$. This cycle plays the role of the fundamental class in topology, hence $LT_{0,k}(d, X)$ is called the virtual fundamental class.

Let V be a convex vector bundle on X. (ie. $H^1(\mathbf{P}^1, f^*V) = 0$ for every holomorphic map $f : \mathbf{P}^1 \to X$.) Then V induces on each $M_{0,k}(d, X)$ a vector bundle V_d, with fiber at $(C, f, x_1, .., x_k)$ given by the section space $H^0(C, f^*V)$. Let b be any multiplicative characteristic class [26]. (ie. if $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles, then $b(E) = b(E')b(E'')$.) The problem we study here is to compute the characteristic numbers

$$K_d := \int_{LT_{0,0}(d,X)} b(V_d)$$

and their generating function:

$$\Phi(t) := \sum K_d\ e^{d \cdot t}.$$

There is a similar and equally important problem if one starts from a concave vector bundle V [37]. (ie. $H^0(\mathbf{P}^1, f^*V) = 0$ for every holomorphic map $f : \mathbf{P}^1 \to X$.) More generally, V can be a direct sum of a convex and a concave bundle. Important progress made on these problems has come from mirror symmetry. All of it seems to point toward the following general phenomenon [12], which we call *the Mirror Principle*. Roughly, it says that the function $\Phi(t)$ can be computed by a change of variables in terms of certain explicit special functions, loosely called generalized hypergeometric functions.

When X is a toric manifold with $c_1(X) \geq 0$, b is the Euler class, and V is a sum of line bundles, there is a general formula derived in [29] from mirror symmetry. This formula was later studied in [21] based on a series of axioms.

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1.1. Main Ideas

We now sketch our main ideas for computing the classes $b(V_d)$.

Step 1. Localization on the linear sigma model. Consider the moduli spaces $M_d(X) := M_{0,0}((1,d),\mathbf{P}^1\times X)$. The projection $\mathbf{P}^1\times X\rightarrow X$ induces a map $\pi: M_d(X)\rightarrow M_{0,0}(d,X)$. Moreover, the standard action of S^1 on $\mathbf{P}^1$ induces an S^1 action on $M_d(X)$. We first study a slightly different problem. Namely consider the classes $\pi^*b(V_d)$ on $M_d(X)$, instead of $b(V_d)$ on $M_{0,0}(d,X)$. First, there is a canonical way to embed fiber products (see below)

$$F_r = M_{0,1}(r,X)\times_X M_{0,1}(d-r,X)$$

each as an S^1 fixed point component into $M_d(X)$. Let $i_r: F_r\rightarrow M_d(X)$ be the inclusion map. Second, there is an evaluation map $e: F_r\rightarrow X$ for each r. Third, suppose that there is a projective manifold W_d with S^1 action, that there is an equivariant map $\varphi: M_d(X)\rightarrow W_d$, and embeddings $j_r: X\rightarrow W_d$, such that the diagram

$$\begin{array}{ccc} F_r & \xrightarrow{i_r} & M_d(X) \\ e\downarrow & & \downarrow\varphi \\ X & \xrightarrow{j_r} & W_d \end{array}$$

commutes. Let α denotes the weight of the standard S^1 action on $\mathbf{P}^1$. Then applying the localization formula [3], this diagram allows us to recast our problem to one of studying the S^1-equivariant classes

$$Q_d := \varphi_!\pi^*b(V_d)$$

defined on W_d. Moreover we can expand the class

$$A_d := \frac{j_0^*Q_d}{e_{S^1}(X_0/W_d)}$$

on X in powers of α^{-1}, and find that it is of order α^{-2}.

The spaces W_d in the commutative diagram above are called the linear sigma model of X. They have been introduced in [39] following [45] when X is a toric manifold,

Step 2. Gluing identity. Consider the vector bundle $\mathcal{U}_d := \pi^*V_d\rightarrow M_d(X)$, restricted to the fixed point components F_r. A point in (C,f) in F_r is a pair $(C_1,f_1,x_1)\times(C_2,f_2,x_2)$ of 1-pointed stable maps glued together at the marked points, ie. $f_1(x_1)=f_2(x_2)$. From this, we get an exact sequence of bundles on F_r:

$$0\rightarrow i_r^*\mathcal{U}_d\rightarrow U_r'\oplus U_{d-r}'\rightarrow e^*V\rightarrow 0.$$

Here $i_r^*\mathcal{U}_d$ is the restriction to F_r of the bundle $\mathcal{U}_d \to M_d(X)$. And U_r' is the pullback of the bundle $U_r \to M_{0,1}(d,X)$ induced by V, and similarly for U_{d-r}'. Taking the multiplicative characteristic class b, we get the identity on F_r:

$$e^*b(V)b(i_r^*\mathcal{U}_d) = b(U_r')b(U_{d-r}').$$

This is what we call the *gluing identity.* This may be translated to a similar quadratic identity, via Step 1, for Q_d in the equivariant cohomology groups $H_{S^1}^*(W_d)$. The new identity is called the Euler data identity.

Step 3. Linking theorem. The construction above is functorial, so that if X comes equipped with a torus T action, then the entire construction becomes $G = S^1 \times T$ equivariant and not just S^1 equivariant. In particular, the Euler data identity is an identity of G-equivariant classes on W_d. Our problem is to first compute the G-equivariant classes Q_d on W_d satisfying the Euler data identity, and with the property that $A_d \sim \alpha^{-2}$. Note that the restrictions $Q_d|_p$ to the T fixed points p in $X_0 \subset W_d$ are polynomial functions on the Lie algebra of G. Suppose that X is a balloon manifold. Then it can be shown that (with a nondegeneracy assumption on $e_G(X_0/W_d)$) the classes Q_d are uniquely determined by the values of the $Q_d|_p$, when α is some scalar multiple of a weight on the tangent space T_pX. These values of $Q_d|_p$ can be computed explicitly by exploiting the structure of a balloon manifold.

Once these values are known, it is often easy to manufacture explicit G-equivariant classes $\tilde{Q}_d$ with the restrictions $\tilde{Q}_d|_p$ having the above same values, and satisfying the Euler data identity. In this case, we say that the data $\tilde{Q}_d$ are linked to the data Q_d. By a suitable change of variables, one can also arrange that $\frac{j_0^*\tilde{Q}_d}{e_{S^1}(X_0/W_d)} \sim \alpha^{-2}$. By the preceding discussion, we get $Q_d = \tilde{Q}_d$.

Step 4. Computing $\Phi(t)$. Once the classes $Q_d = \varphi_!\pi^*b(V_d)$ are determined, we can unwind the many maps used in Step 1. The preceding computations can be done simply in the form of power series. This finally computes the generating function $\Phi(t)$.

The answer for $\Phi(t)$ is given in the form of Conjecture 9.1. In this paper, for clarity, we restrict ourselves to the case when the tangent bundle of X is convex. We prove that Conjecture 9.1 holds whenever X is a balloon manifold having a linear sigma model W_d such that $e_G(X_0/W_d)$ satisfies a nondegeneracy condition.

In the nonconvex case, we must replace $M_{0,k}(d,X)$ by Li-Tian's virtual fundamental cycle [34] for the purpose of localization and integration. The sequel, Mirror Principle III,

to this paper will be devoted entirely to dealing with the added technicality arising from this replacement. All the results in this paper will generalize with only slight modifications as a result of this replacement, but with no change to the overall conceptual framework.

By the equivalence, established in [35], of symplectic GW theory and algebraic GW theory for projective manifolds, we also expect that the results in this paper can be readily generalized to the symplectic case [43][44].

2. Set-up

2.1. Equivariant localization

We first discuss some basic facts about localization. The key technique of our proof is the equivariant localization formula, due to Atiyah-Bott [2][10][3], and Berline-Vergne [9]. For an orbifold version of the localization formula, see [31]. The spirit of the localization we'll use is closer to the Bott residue formula. We first explain this formula.

Let X and Y be two spaces, by which we mean compact manifolds or orbifolds, with a torus T-action. When an orbifold is involved, the integral and localization formulas should be taken in the orbifold sense. Let $\{F\}$ be the components of the fixed point set. Let $H_T^*(\cdot)$ denote the equivariant cohomology group with complex coefficient, and $i_F : F \to X$ the inclusion map. We say that equivariant localization holds on X, if the two maps

$$i_F^* : \quad H_T^*(X) \to H_T^*(F), \quad i_{F!} : \quad H_T^*(F) \to \ H_T^*(X).$$

which are respectively the pull-back, and the Gysin map, are such that the following formulas holds: given any equivariant cohomology class ω on X, we have

$$\omega = \sum_F i_{F!} \left(\frac{i_F^* \omega}{e_T(F/X)} \right).$$

This formula is equivalent to the integral version of the localization formula

$$\int_X \omega = \sum_F \int_F \frac{i_F^* \omega}{e_T(F/X)}.$$

An important fact about equivariant theory is that, if V is an equivariant vector bundle on an orbifold X, then any characteristic class of V has an equivariant extension.

Let $T = S^1$ for simplicity. If c_{2k} is a characteristic class of degree $2k$, then its equivariant extension can be represented by the form

$$c = c_{2k} + c_{2k-2} + \cdots + c_0$$

in the equivariant cohomology of X.

Here is a way to calculate the terms in the localization formula. Assume that each fixed point component F is smooth. If c_{2k} is a Chern class, then by using splitting principle, it can be expressed as a symmetric function of the Chern roots: $P(x_1, \cdots, x_l)$ where $l = rank\ V$. When V can be decomposed into a direct sum of line bundles on F:

$$V|_F = L_1 \oplus \cdots \oplus L_l$$

with the T-action on L_j given by, say the character $e^{2\pi\sqrt{-1}n_j t}$, then the restriction of its equivariant counterpart c to F is

$$i_F^* c = P(c_1(L_1) + n_1 t, \cdots, c_1(L_l) + n_l t).$$

The computation of the equivariant Euler class of F in X is similar. When the restriction of TX to F has a decompostion into line bundles

$$TX|_F = E_1 \oplus \cdots \oplus E_n$$

where T acts on E_j by the character $e^{2\pi\sqrt{-1}m_j t}$, then

$$e_T(F/X) = \prod_j (c_1(E_j) + m_j t).$$

In the above, the n_j's and m_j's are integers for if X is a manifold, and are ratonal numbers if X is an orbifold.

2.2. Functorial localization formula

In this subsection, we derive two formulas which are often used in our work. Let X, Y be two T-spaces, and

$$f: \quad X \to Y$$

be an equivariant map. Let E be a fixed point component in Y, and $F := f^{-1}(E)$ be a fixed component in X. Let g be the restriction of f to F, and $j_E : E \to Y$, $i_F : F \to X$ be the inclusion maps. Thus we have the commutative diagram:

$$\begin{array}{ccc} F & \xrightarrow{i_F} & X \\ g\downarrow & & \downarrow f \\ E & \xrightarrow{j_E} & Y. \end{array}$$

Lemma 2.1. *Given any class $\omega \in H_T^*(X)$, we have the equality on E:*

$$\frac{j_E^* f_!(\omega)}{e_T(E/Y)} = g_! \left(\frac{i_F^* \omega}{e_T(F/X)} \right)$$

Proof: Let us consider localization of $\omega f^* j_{E!}(1)$ on X,

$$\omega f^* j_{E!}(1) = i_{F!} \left(\frac{i_F^*(\omega f^* j_{E!}(1))}{e_T(F/X)} \right).$$

Note the contributions from fixed components other than F vanish. Applying the push-forward $f_!$ to both sides, we get

$$f_!(\omega) j_{E!}(1) = f_! i_{F!} \left(\frac{i_F^*(\omega f^* j_{E!}(1))}{e_T(F/X)} \right).$$

Now $f \circ i_F = j_E \circ g$ which, implies

$$f_! i_{F!} = j_{E!} g_!, \quad i_F^* f^* = g^* j_E^*.$$

Thus we get

$$f_!(\omega) j_{E!}(1) = j_{E!} g_! \left(\frac{i_F^*(\omega)\ g^* e_T(E/Y)}{e_T(F/X)} \right).$$

Applying j_E^* to both sides, we then arrive at

$$j_E^* f_!(\omega)\ e_T(E/Y) = e_T(E/Y)\ g_! \left(\frac{i_F^*(\omega)\ g^* e_T(E/Y)}{e_T(F/X)} \right) = e_T(E/Y)^2\ g_! \left(\frac{i_F^*(\omega)}{e_T(F/X)} \right).$$

Since $e_T(E/Y)$ is invertible, our assertion follows. $\square$

The same argument applies to the case when E and F are T-invariant subspaces. A slightly different argument for the proof of the above lemma will be given in our subsequent

paper. We will also need the following formula, which is actually a special case of [18], Theorem 6.2. Here we include a proof for the convenience of the reader. The spaces involved are T-spaces, that is, T-manifolds or orbifolds.

Lemma 2.2. *Suppose we have a T-equivariant commutative diagram*

$$\begin{array}{ccc} V & \xrightarrow{i} & W \\ g\downarrow & & \downarrow f \\ Z & \xrightarrow{j} & Y \end{array}$$

such that $f^*j_!(1) = i_!(1)$. Then for any class ω on $H_T^*(W)$, we have the following equality on Z:

$$j^*f_!(\omega) = g_!i^*(\omega).$$

Proof: By assumption, we have

$$\omega\ f^*j_!(1) = \omega\ i_!(1) = i_!i^*(\omega).$$

Applying $j^*f_!$ to both sides, on the one hand we get

$$j^*f_!(\omega\ f^*j_!(1)) = j^*(f_!(\omega)\ j_!(1)) = j^*f_!(\omega)\ e_T(Z/Y).$$

On the other hand, we get

$$j^*f_!(i_!i^*(\omega)) = j^*j_!g_!i^*(\omega) = g_!i^*(\omega)\ e_T(Z/Y).$$

Thus our assertion follows. $\square$

The case we will use in this paper is when Z and $V = f^{-1}(Z)$ are both T invariant submanifolds of same codimension, in which the condition in the Lemma clearly holds.

2.3. Balloon manifolds

By a *balloon manifold*, we mean a complex projective T-manifold X with the following properties. There are only finite number of T-fixed points. At each fixed point p, the T-weights on the isotropic representation T_pX are pairwise linearly independent. This class of manifolds were introduced by Goresky-Kottwitz-MacPherson [22]. (We refer the reader to [24] for an excellent exposition.) Throughout this paper, we assume that X *is convex*, ie. $H^1(\mathbf{P}^1, f^*TX) = 0$ for any holomorphic map $f : \mathbf{P}^1 \to X$.

One important property of a balloon n-fold is that at each fixed point p, there are exactly n balloons, ie. T-invariant $\mathbf{P}^1$, each balloon connecting p to one other fixed point q. The induced action on each balloon is the standard rotation with two fixed points p and q. (see [22][25]). We *denote by pq the balloon connecting the fixed points p, q.* Toric manifolds, complex C-spaces and spherical manifolds are examples of balloon manifolds.

We fix a T equivariant embedding of X into the product of projective spaces

$$\mathbf{P}(n) := \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_m}$$

such that the pull-backs of the hyperplane classes $H = (H_1, \cdots, H_m)$ generate $H^2(X, \mathbf{Q})$. We use the same notations for the corresponding equivariant classes of the H's, and their restrictions to X. For $\omega \in H^2(X)$ and $d \in H_2(X)$, we denote their pairing by $\langle \omega, d \rangle$.

For convenience, we introduce the following notations:

$$\begin{aligned} H &= (H_1, .., H_m) \\ H \cdot \zeta = H_\zeta &= H_1\zeta_1 + \cdots + H_m\zeta_m \\ H(p) &= (H_1(p), .., H_m(p)) \\ H_\zeta(p) &= H_1(p)\zeta_1 + \cdots + H_m(p)\zeta_m. \end{aligned}$$

Here $\zeta = (\zeta_1, .., \zeta_m)$ are formal variables. We denote by $K^\vee \subset H_2(X)$ the set of points in $H_2(X, \mathbf{Z})_{free}$ in the dual of the closure of the Kähler cone of X. Since $K^\vee$ is a semigroup in $H_2(X)$, it defines a partial ordering $\succ$ on the lattice $H_2(X, \mathbf{Z})_{free}$. That is, $d \succeq r$ iff $d - r \in K^\vee$. Let $\{H_j^\vee\}$ be the basis dual to the $\{H_j\}$ in $H_2(X)$. If $d \succeq r$ for two classes d, $r \in H_2(X)$, then $d - r = d_1 H_1^\vee + \cdots + d_m H_m^\vee$ for nonnegative integers $d_1, \cdots, d_m$.

We also consider a balloon manifold as a symplectic manifold with a symplectic structure given by $\omega = H_\zeta$ for some generic ζ. By the convexity theorem of Atiyah [2] and Guillemin-Sternberg [23], the image of the moment map μ_ζ in the dual Lie algebra $\mathcal{T}^*$ is a convex polytope, known as the moment polytope. When X is a toric manifold, the moment polytope is known as a Delzant polytope [15]. In this case, it is well-known that the normal fan of this polytope is the defining fan of X.

We say X a multiplicity-free manifold, if for each point p in $\mathcal{T}^*$, the inverse image $\mu_\zeta^{-1}(p)$ is connected.

Lemma 2.3. *Let X be a multiplicity-free balloon manifold, then $H(p) \neq H(q)$ for any two distinct fixed points p and q in X.*

Proof: Let μ_ζ denote the moment map of the T-action on X with respect to the symplectic form $H_\zeta = \langle H, \zeta \rangle$ for a generic choice of $\zeta \in \mathbf{C}^k$. Then the image of $\mu_\zeta : \ X \to \mathcal{T}^*$ is a

convex polytope whose vertices are given the images of the fixed points $\{p\}$. The weights of H_ζ at the fixed points, up to an over all translation, are the same as $\{\mu_\zeta(p)\}$ which are all different. Since X is multiplicity-free, the inverse image of each vertex contains only one fixed point in X. Since ζ is generic, this implies that the $H(p)$'s are distinct at different fixed point. $\square$

We shall assume throughout this paper that $H(p) \neq H(q)$ *for all distinct fixed points* p, q *in* X. Equivalently, if $c(p) = c(q)$ for all $c \in H_T^2(X)$, then $p = q$. This condition is also equivalent to the statement that the moment map with respect to $\omega = H_\zeta$ and the T action is injective to the set of vertices of the moment polytope, when restricted to the fixed points X^T. By the above lemma we know that toric manifolds and compact homogeneous manifolds all satisfy this condition.

When X is a toric n-fold, we have $N = m + n$ T-invariant divisors in X. Let $D_a = c_1(L_a)$, $a = 1, .., N$, be the equivariant first Chern classes of the corresponding equivariant line bundles. These T divisors correspond 1-1 with the one-cones of the defining fan of X [40]. Moreover the fixed points correspond 1-1 with the n-cones. Labelling the n-cones by $p \in X^T$, we have a balloon [40] pq in X iff the n-cones p, q intersect in a codimension one subcone. Since X is smooth, hence the n-cones are regular, there are exactly n balloons pq for each fixed q. One can give a dual description of all these by using the Delzant polytope.

Returning to the general case, suppose that X is a balloon manifold, and that we have equivariant classes $\{D_a\}$ in $H_T^2(X)$ with the following property. At every fixed point p, $D_a(p)$ is either zero or it is a weight on T_pX. Let pq be a balloon in X. The induced T-action on pq is the standard rotation with fixed points p, q. By applying the localization formula on $pq \simeq \mathbf{P}^1$ and the integral $\langle c, [pq] \rangle$, we have

$$c(q) = c(p) + \langle c, [pq] \rangle \lambda$$

for all $c \in H_T^*(X)$, where λ is the weight on the tangent line $T_q(pq)$. Let $\lambda = D_a(q)$. Specializing to $c = D_a$, we get $D_a(q) = D_a(p) + \langle D_a, [pq] \rangle D_a(q)$. This shows that $\langle D_a, [pq] \rangle \neq 0$. For otherwise we would have $D_a(q) = D_a(p) \neq 0$, and this would mean that $D_a(p)$ is a weight on $T_p(po)$ for some edge po running *in the direction* of $D_a(q) = D_a(p)$ from p to o. So we had three vertices lying joined in a line from q to p to o in the moment graph. This would mean that there is a pair of linearly dependent weights on the tangent space T_pX, which can't happen in a balloon manifold. A similar argument shows that $\langle D_a, [pq] \rangle = 1$.

Lemma 2.4. *Let $\omega = H_\zeta$ and $p, q \in X^T$, $r \succ 0$ and λ be a weight on T_qX. If $\omega(q) = \omega(p) + \langle \omega, r\rangle\lambda$ for generic ζ, then p, q are joined by a balloon, $r = [pq]$, and λ is the weight on the tangent line $T_q(pq)$.*

Proof: It suffices to prove that p, q are joined by a balloon. The last two conclusions then follow immediately. Then under the corresponding moment map μ, p, q are mapped to $\omega(p), \omega(q)$ (up to an overall affine transformation), which are distinct because

$$\omega(q) - \omega(p) = \langle \omega, r\rangle\lambda \neq 0. \tag{2.1}$$

Since λ is a weight on T_qX, there is an edge emanating from the point $\omega(q)$ in the direction of λ, ending at some other vertex $\omega(o)$, where qo is a balloon in X. If $\omega(p) \neq \omega(o)$, we would have three distinct vertices of the moment polytope lying on a single line. Thus $\omega(p) = \omega(o)$, which implies that $p = o$. $\square$

Lemma 2.5. *The zero class $\omega = 0$ is the only class in $H_T^*(X)$ with the property that*

$$\int_X \omega \ e^{H_\zeta} = 0$$

for all generic $\zeta \in \mathbf{C}$.

Proof: Suppose

$$\int_X \omega \ e^{H_\zeta} = 0.$$

By localization, we have

$$\sum_{p \in X^T} \frac{\omega(p)}{e_T(p/X)} e^{H_\zeta(p)} = 0.$$

But since the vectors $H(p)$ are all distinct, those exponential functions in ζ are linearly independent over the field $\mathbf{Q}(\mathcal{T}^*)$, implying that $\omega(p) = 0$ for all p. Thus $\omega = 0$. $\square$

2.4. Sigma models

Let X be balloon manifold with a fixed T-equivariant embedding $X \to \mathbf{P}(n)$, as discussed above. We write

$$M_d(X) := M_{0,0}((1, d), \mathbf{P}^1 \times X).$$

Since X is assumed to be convex, $M_d(X)$ is an orbifold. The standard S^1 action on $\mathbf{P}^1$ together with the T action on X induce a $G = S^1 \times T$ action on $M_d(X)$.

Here is a description of some S^1 fixed point components F_r, labelled by $0 \preceq r \preceq d$, inside of $M_d(X)$. Let F_r be the fiber product

$$F_r := M_{0,1}(r, X) \times_X M_{0,1}(d-r, X)$$

More precisely, consider the map

$$ev_r \times ev_{d-r} : M_{0,1}(r, X) \times M_{0,1}(d-r, X) \to X \times X$$

given by evaluations at the marked points; and

$$\Delta: \ X \to X \times X$$

the diagonal map. Then

$$F_r = (ev_r \times ev_{d-r})^{-1}\Delta(X).$$

Note that $F_d = M_{0,1}(d, X)$ by convention. The set F_r can be identified with an S^1 fixed point component of $M_d(X)$ as follows. Consider the case $r \neq 0, d$ first. Given a point $(C_1, f_1, x_1) \times (C_2, f_2, x_2)$ in F_r, we get a new curve C by gluing C_1, C_2 to $\mathbf{P}^1$ with x_1, x_2 glued to $0, \infty \in \mathbf{P}^1$ respectively. The new curve C is mapped into $\mathbf{P}^1 \times X$ as follows. Map $\mathbf{P}^1 \subset C$ identically onto $\mathbf{P}^1$, and collapse C_1, C_2 to $0, \infty$ respectively; then map C_1, C_2 into X with f_1, f_2 respectively, and collapse the $\mathbf{P}^1$ to $f(x_1) = f(x_2)$. This defines a point (C, f) in $M_d(X)$. For $r = 0$, we glue (C_1, f_1, x_1) to $\mathbf{P}^1$ at x_1 and 0. For $r = d$, we glue (C_2, f_2, x_2) to $\mathbf{P}^1$ at x_2 and ∞. We will identify F_r as a subset of $M_d(X)$ as above, and let

$$i_r : F_r \to M_d(X)$$

denotes the inclusion map. Clearly, we also have an evaluation map

$$e_r : F_r \to X$$

which sends a pair in F_r to the value at the marked point. In the following, we will simply write e_r as e without causing any coonfusion.

We call a compact manifold or orbifold W_d with $G = S^1 \times T$ action a *linear sigma model* of degree d for X, if the following conditions are satisfied:

1. The S^1 action on W_d has fixed point components given by X_r, labelled by $0 \preceq r \preceq d$, and each X_r is T-equivariantly isomorphic to X.

2. There is a G-equivariant birational map φ from $M_d(X)$ to W_d, such that $\varphi|_{F_r} = e$, and $\varphi^{-1}(X_r) = F_r$.
3. All equivariant cohomology classes in $H_G^2(W_d)$ are lifted from $H_T^2(X)$, and the lift $\hat{D} \in H_G^2(W_d)$ of $D \in H_T^2(X)$ restricts to $D + \langle D, r\rangle\alpha$ on X_r.
4. The G-equivariant Euler class of the normal bundle of X_0 in W_d has the form

$$e_G(X_0/W_d) = \prod_a \prod_{m_a} (D_a - m_a\alpha)$$

where the m_a's are positive integers and the D_a's are classes in $H_T^2(X)$, such that at a given T fixed point p in X, the nonzero $D_a(p)$'s are multiples of distinct weights of T_pX.

Here a birational map, in algebraic geometry language, is a regular morphism which is an isomorphism when restricted to a Zariski open set in $M_d(X)$.

Note W_d need not be unique. We identify X_r with X by assumption 1, and denote by

$$j_r : X_r \to W_d$$

the inclusion map.

We call a balloon manifold X *admissible* if it has a linear sigma model W_d for each d, and that $H_\zeta(p) \neq H_\zeta(q)$ for any two distinct fixed points p, q in X. The main result in this paper is to show that the mirror principle holds for any admissible balloon manifold.

Remark 2.6. *Condition 4 is actually assuming more than what we need. This condition can be replaced by the following weaker, but more technical condition. For each fixed point p and for any d, as a function of α, $e_G(X_0/W_d)|_p$ has possible zero only at either 0 or a multiple of a weight λ on T_pX. In addition if $[pq]$ is a balloon and $d = \delta[pq]$, then λ/δ is at worse a simple zero. For example, the following form would meet this criterion:*

$$e_G(X_0/W_d) = \frac{\prod_a \prod_{m_a} (D_a - m_a\alpha)}{\prod_b \prod_{n_b} (D_b - n_b\alpha)}$$

where the m_a, n_b are nonzero scalars.

Example 1: Projective space $\mathbf{P}^n$ with $W_d = \mathbf{P}^{(n+1)d+n}$ is admissible. The existence of φ was proved in [37], which is the so-called Li-Tian map. The lifted hyperplane class κ has the required property that

$$j_r^*\kappa = H + \langle H, r\rangle\alpha = H + r\alpha.$$

The equivariant Euler class

$$e_G(\mathbf{P}_0^n/W_d) = \prod_{i=0}^{n}\prod_{m=1}^{d}(H - \lambda_i - m\alpha)$$

where λ_i's denote the weights of the torus T action on $\mathbf{P}^n$. Clearly the equivariant classes $\{H - \lambda_i\}$ has the required property.

Example 2: More generally for $\mathbf{P}(n)$, we can take $N_d(\mathbf{P}(n))$ to be W_d. In fact the S^1 fixed point components on $N_{k,l}$ are exactly $k+1$ copies $\mathbf{P}^l_r$, $r = 0, .., k$, of $\mathbf{P}^l$. Each $\mathbf{P}^l_r$ consists of $l+1$ tuples of monomials, each being a scalar multiple of $w_0^r w_1^{k-r}$. Similarly the S^1 fixed point components on $N_d(\mathbf{P}(n))$ are copies $\mathbf{P}(n)_r$, $0 \preceq r \preceq d$, of $\mathbf{P}(n)$. All equivariant cohomology classes in $H^2_G(N_d(\mathbf{P}(n))$ are lifted from $H^2_T(\mathbf{P}^n)$ (cf [37]). Let κ_i be the lift of the hyperplane class H_i, of the ith factor $\mathbf{P}^{n_i}$. Then

$$j_r^*\kappa_i = H_i + \langle H_i, r\rangle\alpha$$

where j_r denotes the inclusion of $\mathbf{P}(n)_r$ in $N_d(\mathbf{P}(n))$. By using the formula in [37], it is easy to show the equivariant Euler class $e_G(\mathbf{P}(n)_0/N_d(\mathbf{P}(n)))$, which is a product of e_G's in last example, has the required property.

Example 3: Let $N_{k,l}$ be the space of $l+1$ tuples $[f_0, .., f_l]$ of degree k polynomials $f_i(w_0, w_1)$, modulo scalar. Thus $N_{k,l} \cong \mathbf{P}^{(l+1)k+l}$. It is called the linear sigma model for $\mathbf{P}^l$. (See [37].) Let

$$N_d(\mathbf{P}(n)) := N_{d_1,n_1} \times \cdots \times N_{d_m,n_m}.$$

Recall that we have a collapsing map $\varphi : M_k(\mathbf{P}^l) \to N_{k,l}$, which is $G := S^1 \times T$ equivariant. By taking composite with the projection from $M_d(\mathbf{P}(n))$ to each $M_{d_j}(\mathbf{P}^{n_j})$, we obtain a G-equivariant map

$$M_d(\mathbf{P}(n)) \to N_d(\mathbf{P}(n))$$

which we also denote by φ. Note that $M_d(X)$ can be viewed as a cycle in $M_d(\mathbf{P}(n))$. We denote the image cycle $\varphi_!(M_d(X))$ in $N_d(\mathbf{P}(n))$ φ by $N_d(X)$.

If $N_d(X)$ is a manifold or an orbifold, then Properties 1-3 are automatically satisfied, if furthermore $e_G(X_0/W_d)$ has property 4, then we can simply take $W_d = N_d(X)$ as the linear sigma model.

Example 4: Convex toric varieties. In this case W_d is a toric n-fold, as introduced by Witten [45]and used first by Morrison-Plesser [39]to study quantum cohomology. Recall that a toric manifold X can be realized as the GIT quotient $\mathbf{C}^N//T^m_{\mathbf{C}}$ where $T^m_{\mathbf{C}}$ is a m-dimensional complex torus acting on $\mathbf{C}^N$. Here $m = rank\ H^2(X, \mathbf{Z})$, $N = n + m$. Let $[z_1, \cdots, z_N]$ denote the coordinates on $\mathbf{C}^N$. Then each z_j can be viewed as a section of a line bundle L_j on X [14][39]. Modulo the induced action by T^m_C from $\mathbf{C}^N$, a map from $\mathbf{P}^1$ into X is uniquely represented by an N-tuple of polynomials

$$[f_1(w_0, w_1), \cdots, f_N(w_0, w_1)]$$

where f_j is a section of the line bundle $O(l_j)$ over $\mathbf{P}^1$ with $l_j = \langle c_1(L_j), d\rangle$. Let $\mathbf{C}^N(d)$ be the vector space of N-tuple of polynomials of degree $(d_1, \cdots, d_N)$ as above. Then as described in [39], W_d is the GIT quotient by the induced action of T^m_C on it:

$$W_d = \mathbf{C}^N(d)//T^m_C.$$

Let $M^o_d(X)$ denote the set of points (f, C) in $M_d(X)$ such that $C \simeq \mathbf{P}^1$. We call $M^o_d(X)$ the smooth part of $M_d(X)$. We can define a map φ_o from $M^o_d(X)$ to W_d in the following way: each (f, C) gives a map from $\mathbf{P}^1$ to X, and modulo the induced T^m_C action, uniquely determines N-tuple of polynomials as above, therefore gives a point in W_d, which we define to be the image of (f, C) under φ_o. This is clearly a canonical identification.

It is not difficult to see that the S^1-fixed components in W_d can be described as GIT quotient,

$$X_r \simeq \{[a_1 w_0^{\langle D_0, r\rangle} w_1^{\langle D_0, d-r\rangle}, \cdots, a_N w_0^{\langle D_N, r\rangle} w_1^{\langle D_N, d-r\rangle}] | a \in \mathbf{C}^N\}//T^m_C.$$

The equivariant Euler class of its normal bundle in W_d is

$$e_G(X_r/W_d) = \prod_{a=1}^{N} \prod_{k=0, k\neq\langle D_a, r\rangle}^{\langle D_a, d\rangle} (D_a + \langle D_a, r\rangle\alpha - k\alpha).$$

Here $D_a = c_1(L_a)$ is the equivariant first Chern class of the line bundle L_a corresponding to the ath component in the coordinates of X. The lift $\hat{D}_a$ of D_a to W_d clearly has the property

$$j_r^* \hat{D}_a = D_a + \langle D_a, r\rangle\alpha.$$

As pointed out in [39], the cohomology of W_d are generated by the $\hat{D}_a$. Thus W_d has properties 1, 3, and 4. In the next subsection, we establish that the φ_o extends to a regular G-equivariant map φ from $M_d(X)$ to W_d, with property 2. So, for a toric X we can take its linear sigma model to be W_d as constructed above. It follows that X is an admissible balloon manifold.

Example 5: Our method works well even for certain singular manifolds. We take weighted projective space as example to illustrate the ideas. Let $\mathbf{P}^n_a$ with $a = (a_0, \cdots, a_n)$ be a weighted projective space. Let $[z_0, \cdots, z_n]$ be the coordinates for $\mathbf{P}^n_a$, then z_j can be considered as a section of the line bundle $O(a_j)$. Then the linear sigma model for this weighted projective space is the induced weighted quotient by S^1 on the space of $n+1$ tuple of polynomials $[f_0(w_0, w_1), \cdots, f_1(w_0, w_1)]$ where f_j is a section of the line bundle $O(da_j)$ on $\mathbf{P}^1$.

It is known that $\mathbf{P}^n_a$ is equivalent to $\mathbf{P}^n/Z_a$ where Z_a is a finite group. The space $M_d(\mathbf{P}^n_a)$ for $\mathbf{P}^n_a$ is equal to $M_d(\mathbf{P}^n)/Z_a$. On the other hand we can also take $N_d(\mathbf{P}^n)/Z_a$ as the linear sigma model W_d. In this case W_d is an orbifold, a weighted projective space. Since the action of Z_a commutes with the action of torus T, we see that the induced collapsing map

$$\varphi : \quad M_d(\mathbf{P}^n_a) \to N_d(\mathbf{P}^n_a) = W_d$$

is clearly a regular map. The corresponding equivariant Euler class has the expression:

$$e_G(\mathbf{P}^n_a/W_d) = \prod_{j=0}^{n} \prod_{m=1}^{da_j} (a_j H - \lambda_j - m\alpha)$$

with H the T-equivariant hyperplane class and λ_j's the weights of the T-action.

Examples of singular toric varieties will be discussed again in our subsequent paper, in which resolution of singularities will be used to reduce to the smooth case. The above example was motivated by a question of Mazur, who suggested that the situation of counting rational curves in orbifolds is similar to certain Diophantine problem in number theory.

Example 6: For a general projective manifold X embedded in $\mathbf{P}(n)$, assume it is defined by a system of polynomial equations $P(z^1, \cdots, z^n) = 0$ where $z^j = (z^j_1, \cdots, z^j_{n_j})$ denotes the coordinate of $\mathbf{P}^{n_j}$. Assume the variety defined by the induced equation $P(f^1, \cdots, f^n) = 0$ in $N_d(\mathbf{P}(n))$ where $f^j = (f^j_1(w_0, w_1), \cdots, f^j_{n_j}(w_0, w_1))$ is the tuple of polynomials, the coordinates for the linear sigma model $N_d(\mathbf{P}^{n_j})$, is an orbifold. Then we

can take it to be our linear sigma model W_d. Note that $N_d(X)$ in Example 3 is embedded inside this W_d. Very likely they are the same.

Though we don't know whether this variety is an orbifold or not, it is clear that the fixed point compoenents in the above variety are given by X_r's. In fact, we only need to assume that the localization formula holds on it. This is the case if the fixed point components embedded into W_d as local complete intersection subvarieties. We conjecture that this is the case for any convex projective manifold. Later, we will state a general conjectural Mirror Formula in terms of this W_d.

2.5. Regularity of the collapsing map

For a toric manifold X, the following lemma show that W_d described in Example 4 is a linear sigma model of X.

Lemma 2.7. *For toric manifold X, there is a regular extension*

$$\varphi : \ M_d(X) \to W_d$$

of the map φ_o in Example 4 above.

Proof: We simply follow the argument in [37], together with the construction in [14]. We will define a morphism $\varphi : M_d(X) \to W_d$. Let $\mathcal{S}$ be the category of all schemes of finite type (over $\mathbf{C}$) and let

$$\mathcal{F} : \mathcal{S} \longrightarrow (\mathrm{Set})$$

be the the contra-variant functor that send any $S \in \mathcal{S}$ to the set of families of stable morphisms

$$F : \mathcal{X} \longrightarrow \mathbf{P}^1 \times X \times S$$

over S, where $\mathcal{X}$ are families of connected arithmetic genus 0 curves, modulo the obvious equivalence relation. Note that $\mathcal{F}$ is represented by the moduli stack $M_d(X)$. Hence to define the morphism φ, it suffices to define a transformation

$$\Psi : \mathcal{F} \longrightarrow \mathrm{Mor}\,(-, W_d).$$

We now define such a transformation. Let $S \in \mathcal{S}$ and let $\xi \in \mathcal{F}(S)$ be represented by $(\mathcal{X}, F)$. We let p_i be the composite of F with the i-th projection of $\mathbf{P}^1 \times X \times S$ and let

p_{ij} be the composite of F with the projection from $\mathbf{P}^1 \times X \times S$ to the product of its i-th and j-th components. We consider the sheaf $p_2^* \mathcal{O}_X(L_j)$ on $\mathcal{X}$ and its direct image sheaf

$$\mathcal{L}_{j,\xi} = p_{13*} p_2^* \mathcal{O}_X(L_j).$$

Here the L_j are the line bundles on X, as defined in Example 4. As in [37], one can show that $\mathcal{L}_{j,\xi}$ is flat in a standard way.

For the same reasoning, the sheaves $\mathcal{L}_{j,\xi}$ satisfy the following base change property: let $\rho : T \to S$ be any base change and let $\rho^*(\xi) \in \mathcal{F}(T)$ be the pull back of ξ. Then there is a canonical isomorphism of sheaves of $\mathcal{O}_T$-modules

$$\mathcal{L}_{j,\rho^*(\xi)} \cong (\mathbf{1}_{\mathbf{P}^1} \times \rho)^* \mathcal{L}_{j,\xi}. \tag{2.2}$$

Since $\mathcal{L}_{j,\xi}$ is flat over S, we can define the determinant line bundle of $\mathcal{L}_{j,\xi}$, denoted by $\det(\mathcal{L}_{j,\xi})$ which is an invertible sheaf over $\mathbf{P}^1 \times S$. Using the Riemann-Roch theorem, one finds that its degree along fibers over S is $l_j = \langle c_1(L_j), d \rangle$. Furthermore, because $\mathcal{L}_{j,\xi}$ has rank one, there is a canonical homomorphism

$$\mathcal{L}_{j,\xi} \longrightarrow \det(\mathcal{L}_{j,\xi}), \tag{2.3}$$

so that its kernel is the torsion subsheaf of $\mathcal{L}_{j,\xi}$.

Let z_j be the j-th homogeneous coordinate of X (Example 4). Then z_j is a section in $H^0(X, L_j)$. Its pull-back is a section of $\mathcal{L}_{j,\xi}$, which induces a section

$$\sigma_{j,\xi} \in H^0(S, \pi_{S*} \det(\mathcal{L}_{j,\xi})).$$

based on (2.3). Then after fixing an isomorphism

$$\det(\mathcal{L}_{j,\xi}) \cong \pi_S^* \mathcal{M} \otimes \pi_{\mathbf{P}^1}^* \mathcal{O}_{\mathbf{P}^1}(l_j) \tag{2.4}$$

for some invertible sheaf $\mathcal{M}$ of $\mathcal{O}_S$-modules, where $l_j = \langle c_1(L_j), d \rangle$. We then obtain a section in

$$\pi_{S*}(\pi_{\mathbf{P}^1}^* \mathcal{O}_{\mathbf{P}^1}(l_j)) \otimes_{\mathcal{O}_S} \mathcal{M} \equiv H^0_{\mathbf{P}^1}(\mathcal{O}_{\mathbf{P}^1}(l_j)) \otimes_{\mathbf{C}} \mathcal{M}.$$

So $\sigma_{j,\xi}$ is determined up to certain constant λ_j coming from $\mathcal{M}$.

Now apply the above argument to each L_j, $j = 1, \cdots, N$, we get N sections $[\sigma_{1,\xi}, \cdots, \sigma_{N,\xi}]$. Let w_0, w_1 be the homogeneous coordinate of $\mathbf{P}^1$, we will write $\sigma_{j,\xi} = f_j(w_0, w_1)$ as a homogeneous polynomial of degree l_j. In this way we get a point in $\mathbf{C}^N(d)$.

The constants λ_j from $\mathcal{M}$ in choosing $\sigma_{j,\xi}$ must satisfy the relation $\prod_j \lambda_j^{\langle m,n_j\rangle} = 1$. Here the n_j are vectors in an integral lattice, which generate the 1-cones in the defining fan of X, and m is any element in the dual lattice. (See [14].)

For such λ_j's we can then find an element g in T_C^m such that $[\lambda_1 f_1, \cdots, \lambda_N f_N]$ is transformed to $[f_1, \cdots, f_N]$ by g, therefore they represent the same point in W_d. In this way, after taking GIT quotient by T_C^m, the induced action on $[f_1(w_0, w_1), \cdots, f_N(w_0, w_1)]$ from the action on $\mathbf{C}^N$, for each map $(f, C) \in M_d(X)$, we have obtained canonically a point in W_d, therefore a morphism

$$\Psi(S) : S \longrightarrow W_d$$

that is independent of the isomorphisms (2.4). It follows from the base change property (2.2) that the collection $\Psi(S)$ defines a transformation

$$\boldsymbol{\Psi} : \mathcal{F} \longrightarrow \mathrm{Mor}\,(-, W_d),$$

thus defines the morphism φ as desired.

The fact that $\varphi : M_d(X) \to W_d$ is $S^1 \times T^N$-equivariant follows immediately from the fact that φ is induced by the transformation Ψ of functors. This completes the proof. $\square$

For another proof of the above lemma we can proceed as follows. We use the notations as in the above Example 3. We show that the regularity of the collapsing map for $\mathbf{P}(n)$ induces the regularity of the collapsing map for X. For this we only need to prove that $N_d(X)$, the image $\varphi(M_d(X))$ in $N_d(\mathbf{P}(n))$ of the collapsing map for $\mathbf{P}(n)$, lies in W_d.

First, we show that W_d lies in $N_d(\mathbf{P}(n))$. Note that both W_d and $N_d(\mathbf{P}(n))$ are toric manifolds, and a Zariski open subset W_d^o in W_d is embedded G-equivariantly in $N_d(\mathbf{P}(n))$. Also the G-fixed points in W_d are all in X_r, therefore in $\mathbf{P}(n)_r$ and $N_d(P(n))$. Any point in W_d is in the closure of a generic $G_{\mathbf{C}}$ orbit in W_d passing through two G fixed points in X_r's. By the equivariance, this $G_{\mathbf{C}}$ orbit is also in $N_d(\mathbf{P}(n))$, therefore the closure of this orbit lies in $N_d(\mathbf{P}(n))$.

Second, we show that $N_d(X)$ lies in W_d. For this we note that φ_o extends to $M_d(X)$, since it is actually the restriction of the corresponding map on $M_d(\mathbf{P}(n))$. Now by taking closure of the inclusion $\varphi(M_d^o(X)) \subseteq W_d$, which is induced from the canonical identification, we get

$$\varphi(M_d(X)) = \overline{\varphi(M_d^o(X))} \subseteq \overline{W_d} = W_d,$$

since W_d is itself closed.

3. The Gluing Identity

Returning to the general case, we let X be an admissible balloon manifold from now on. In this section, we apply the functorial localization formula to the linear sigma model. The argument used here is modelled on the one used in [37], except that the T-action is *not* used here. Thus all the results in this section hold for manifolds without T action. We will have more to say about the mirror principle without T action later.

Recall that we have the commutative diagram:

$$\begin{array}{ccc} F_r & \stackrel{i_r}{\longrightarrow} & M_d(X) \\ e\downarrow & & \downarrow\varphi \\ X_r & \stackrel{j_r}{\longrightarrow} & W_d. \end{array}$$

We also have the natural forgetting map $\rho : M_{0,1}(d,X) \to M_{0,0}(d,X)$, and the projection map $\pi : M_d(X) \to M_{0,0}(d,X)$. Note that we have a commutative diagram

$$\begin{array}{ccc} M_d(X) & & \\ \pi\downarrow & \nwarrow i_0 & \\ M_{0,0}(d,X) & \stackrel{\rho}{\longleftarrow} & M_{0,1}(d,X). \end{array}$$

Let $\varphi : M_d(X) \to W_d,\ e : F_r \to X_r$ play the respective roles of $f : X \to Y,\ g : F \to E$ in the functorial localization formula. Then it follows that

Lemma 3.1. *Given any G-equivariant cohomology class ω on $M_d(X)$, we have the following equality on X_r for $0 \preceq r \preceq d$:*

$$\frac{j_r^*\varphi_!(\omega)}{e_G(X_r/W_d)} = e_!\left(\frac{i_r^*(\omega)}{e_G(F_r/M_d(X))}\right).$$

Actually this lemma may be viewed as an equivariant version of the so-called excess intersection formula of [18], Theorem 6.3.

Let L_r denote the line bundle on $M_{0,1}(r,X)$ whose fiber at $(f,C;x)$ is the tangent line at the marked point $x \in C$. Let π_1 denote the projection from $\mathbf{P}^1 \times X$ to $\mathbf{P}^1$.

The normal bundle of F_r in $M_d(X)$ can be computed just as in [37]. For $r \neq 0, d$, we have

$$N(F_r/M_d(X)) = H^0(C_0, (\pi_1 \circ f)^* T\mathbf{P}^1) + T_{x_1}C_0 \otimes L_r + T_{x_2}C_0 \otimes L_{d-r} - A_{C_0}.$$

Here we have used the notations as in [37]: a point (f_1, C_1, x_1) in $M_{0,1}(r, X)$ and a point (f_2, C_2, x_2) in $M_{0,1}(d-r, X)$ is glued to $C_0 \simeq \mathbf{P}^1$ at 0 and ∞ respectively to get the point (f, C) in $M_d(X)$ with $C \simeq C_1 \cup C_0 \cup C_2$. Since x_1 and x_2 are mapped to the same point in X under the projection $\pi_2: \mathbf{P}^1 \times X \to X$, so this point can be considered as a point in F_r by gluing together (f_1, C_1, x_1) and (f_2, C_2, x_2) at the marked points. Similarly, for $r = 0, d$, we have

$$N(F_0/M_d(X)) = H^0(C_0, (\pi_1 \circ f)^* T\mathbf{P}^1) + T_{x_1} C_0 \otimes L_d - A_{C_0}$$

and

$$N(F_d/M_d(X)) = H^0(C_0, (\pi_1 \circ f)^* T\mathbf{P}^1) + T_{x_2} C_0 \otimes L_d - A_{C_0}.$$

In the above $H^0(C_0, (\pi_1 \circ f)^* T\mathbf{P}^1)$ corresponds to the deformation of C_0; $T_{x_1} C_0 \otimes L_r$ and $T_{x_2} C_0 \otimes L_{d-r}$ correspond respectively to the deformations of the nodal points x_1 and x_2; A_{C_0} denotes the automorphism group to be quotiented out.

The equivariant Euler classes of the normal bundles above are computed as in [37], to which we refer the readers for details. For $r \neq 0, d$, the equivariant Euler classes are:

$$e_G(F_r/M_d(X)) = -\alpha(-\alpha + c_1(L_{d-r})) \cdot \alpha(\alpha + c_1(L_r))$$

where the two factors on the right hand side are pullbacked to F_r from $M_{0,1}(d-r, X)$, $M_{0,1}(r, X)$ respectively. For $r = 0, d$, we have

$$e_G(F_0/M_d(X)) = -\alpha(-\alpha + c_1(L_d)), \;\; e_G(F_d/M_d(X)) = \alpha(\alpha + c_1(L_d))$$

respectively. Combining this with the preceding lemma, we get the following equality on $X = X_0$:

$$\frac{j_0^* \varphi_!(\omega)}{e_G(X_0/W_d)} = ev_! \left(\frac{i_0^*(\omega)}{\alpha(\alpha - c_1(L_d))} \right).$$

Here we have dropped the subscript from ev_d. In particular, if ψ is a class on $M_{0,0}(d, X)$, then for $\omega = \pi^* \psi$, we get $i_0^*(\omega) = i_0^*(\pi^* \psi) = \rho^* \psi$. This yields

Lemma 3.2. *Given any T-equivariant cohomology class ψ on $M_{0,0}(d, X)$, we have the following equality on X:*

$$\frac{j_0^* \varphi_!(\pi^* \psi)}{e_G(X_0/W_d)} = ev_! \left(\frac{\rho^* \psi}{\alpha(\alpha - c_1(L_d))} \right).$$

Lemma 3.3. *For $0 \preceq r \preceq d$, we have the following equality on X:*

$$e_G(X_r/W_d) = \overline{e_G(X_0/W_r)}e_G(X_0/W_{d-r}).$$

In particular, we have

$$e_G(X_d/W_d) = \overline{e_G(X_0/W_d)}.$$

Proof: Consider the commutative diagram

$$\begin{array}{ccc} F_r & \xrightarrow{\Delta_0} & M_{0,1}(r,X) \times M_{0,1}(d-r,X) \\ e \downarrow & & \downarrow ev_r \times ev_{d-r} \\ X & \xrightarrow{\Delta} & X \times X \end{array} \tag{3.1}$$

where Δ is the diagonal map, and Δ_0 is the inclusion induced by Δ. In particular, by definition we have $(ev_r \times ev_{d-r})^* \Delta_!(1) = (\Delta_0)_!(1)$. So we have

$$\Delta^*(ev_r \times ev_{d-r})_!(\omega) = e_! \Delta_0^*(\omega) \tag{3.2}$$

for any class ω on $M_{0,1}(r,X) \times M_{0,1}(d-r,X)$. Now put $\omega = \frac{1}{\alpha(\alpha+c_1(L_r))} \times \frac{1}{\alpha(\alpha-c_1(L_{d-r}))}$. Then (3.2) becomes

$$(ev_r)_! \frac{1}{\overline{e_G(F_0/M_r(X))}} \cdot (ev_{d-r})_! \frac{1}{e_G(F_0/M_{d-r}(X))} = e_! \frac{1}{e_G(F_r/M_d(X))}.$$

Since $\varphi : M_d(X) \to W_d$ is an isomorphism on a Zariski open set, we see that $\varphi_!(1) = 1$. In fact, by Prop. (5.3.3) [1], $\varphi_!$ preserves degree because since $M_d(X)$ and W_d have the same dimension. So $\varphi_!(1) \in H^0_G(W_d)$ is a constant. By restricting to the Zariski open set on which φ is an isomorphism, we find $\varphi_!(1) = 1$.

By taking $\psi = 1$ in the preceding lemma, we get

$$(ev_r)_! \frac{1}{e_G(F_0/M_r(X))} = \frac{1}{e_G(X_0/W_r)}, \quad (ev_{d-r})_! \frac{1}{e_G(F_0/M_{d-r}(X))} = \frac{1}{e_G(X_0/W_{d-r})}.$$

By taking $\omega = 1$ in Lemma 3.1, we get

$$e_! \frac{1}{e_G(F_r/M_d(X))} = \frac{1}{e_G(X_r/W_d)}.$$

Combining the last four equations yields our assertion. $\square$

Fix a T-equivariant multiplicative class b_T. Fix a T-equivariant bundle of the form $V = V^+ \oplus V^-$, where $V^\pm$ are respectively the convex/concave bundles. (cf. [37].) We call such a V a mixed bundle. We assume that

$$\Omega := \frac{b_T(V^+)}{b_T(V^-)}$$

is a well-defined invertible class on X. By convention, if $V = V^\pm$ is purely convex/concave, then $\Omega = b_T(V^\pm)^{\pm 1}$. Recall that the bundle $V \to X$ induces the bundles

$$V_d \to M_{0,0}(d, X), \quad U_d \to M_{0,1}(d, X), \quad \mathcal{U}_d \to M_d(X).$$

Moreover, they are related by $U_d = \rho^* V_d$, $\mathcal{U}_d = \pi^* V_d$, Throughout this section, we denote

$$Q: \quad Q_d := \varphi_!(\pi^* b_T(V_d)).$$

If ω is a class on W_d, we write

$$i_r^* \omega^v := \frac{j_r^* \omega}{e_G(X_r/W_d)}$$

which is a class on $X = X_r$.

Lemma 3.4. *For $0 \preceq r \preceq d$,*

$$\Omega \; i_r^* Q_d^v = \overline{i_0^* Q_r^v} \; i_0^* Q_{d-r}^v.$$

Proof: For simplicity, let's consider the case $V = V^+$. The general case is entirely analogous.

Recall that a point (f, C) in $F_r \subset M_d$ comes from gluing together a pair of stable maps $(f_1, C_1, x_1), (f_2, C_2, x_2)$ with $f_1(x_1) = f_2(x_2) = p \in X$. From this, we get an exact sequence over C:

$$0 \to f^* V \to f_1^* V \oplus f_2^* V \to V|_p \to 0.$$

Passing to cohomology, we have

$$0 \to H^0(C, f^* V) \to H^0(C_1, f_1^* V) \oplus H^0(C_2, f_2^* V) \to V|_p \to 0.$$

Hence we obtain an exact sequence of bundles on F_r:

$$0 \to i_r^*\mathcal{U}_d \to U'_r \oplus U'_{d-r} \to e^*V \to 0.$$

Here $i_r^*\mathcal{U}_d$ is the restriction to F_r of the bundle $\mathcal{U}_d \to M_d(X)$. And U'_r is the pullback of the bundle $U_r \to M_{0,1}(d,X)$, and similarly for U'_{d-r}. Taking the multiplicative characteristic class b_T, we get the identity on F_r:

$$e^*b_T(V)b_T(i_r^*\mathcal{U}_d) = b_T(U'_r)b_T(U'_{d-r}).$$

This is what we call the *gluing identity.*

Now put

$$\omega = \frac{b_T(U_r)}{e_G(F_r/M_r(X))} \times \frac{b_T(U_{d-r})}{e_G(F_0/M_{d-r}(X))}.$$

From the commutative diagram (3.1), we have the identity:

$$\Delta^*(ev_r \times ev_{d-r})_!(\omega) = e_!\Delta_0^*(\omega).$$

On one hand is

$$\begin{aligned}
\Delta^*(ev_r \times ev_{d-r})_!(\omega) &= (ev_r)_!\frac{b_T(U_r)}{e_G(F_r/M_r(X))} \cdot (ev_{d-r})_!\frac{b_T(U_{d-r})}{e_G(F_0/M_{d-r}(X))} \\
&= (ev_r)_!\frac{\rho^*b_T(V_r)}{e_G(F_r/M_r(X))} \cdot (ev_{d-r})_!\frac{\rho^*b_T(V_{d-r})}{e_G(F_0/M_{d-r}(X))} \\
&= \overline{i_0^*Q_r^v}\ i_0^*Q_{d-r}^v,
\end{aligned}$$

the last equality being a consequence of Lemma 3.2. On the other hand, applying the gluing identity, we have

$$\begin{aligned}
e_!\Delta_0^*(\omega) &= e_!\left(\frac{b_T(U'_r)}{\alpha(\alpha + c_1(L_r))}\ \frac{b_T(U'_{d-r})}{\alpha(\alpha - c_1(L_{d-r}))}\right) \\
&= e_!\frac{e^*b_T(V)i_r^*b_T(\mathcal{U}_d)}{e_G(F_r/M_d(X))} \\
&= b_T(V)\ e_!\frac{i_r^*b_T(\mathcal{U}_d)}{e_G(F_r/M_d(X))} \\
&= b_T(V)i_r^*Q_d^v,
\end{aligned}$$

the last equality being a consequence of Lemma 3.1. This proves our assertion. □

Remark 3.5.

(a) If we take V to be the trivial line bundle, and b_T to be the total Chern class, then the preceding lemma reduces to Lemma 3.3.

(b) All the lemmas in this section, in fact, holds for a general projective manifold X without T-action, provided that we still have the S^1-equivariant map $\varphi : M_d(X) \to W_d$, with properties 1.-2. stated in section 2. All G-equivariant classes above are then replaced by their S^1-equivariant counterparts.

4. Euler Data

Notations: We denote by κ_i the G-equivariant class on W_d with the property that $j_r^*\kappa_i = H_i + \langle H_i, r\rangle\alpha$. By the localization theorem, κ_i is determined by these restriction conditions, and is a class in the localized equivariant cohomology of W_d. More generally a class $\phi \in H_T^2(X)$ has a G-equivariant lift $\hat\phi \in H_G^2(W_d)$ determined by $j_r^*\hat\phi = \phi + \langle\phi, r\rangle\alpha$. We denote by $\langle H_T^2(X)\rangle$ the ring generated by $H_T^2(X)$, and by R_d the ring generated by their lifts $\hat\phi$. We put $\mathcal{R} = \mathbf{Q}(\mathcal{T}^*)[\alpha]$, where $\mathbf{Q}(\mathcal{T}^*)$ is the rational function field on the Lie algebra of T. For convenience, we introduce the notations

$$\kappa\cdot\zeta = \kappa_\zeta := \kappa_1\zeta_1 + \cdots + \kappa_m\zeta_m$$
$$i_r^*\omega^v := \frac{j_r^*\omega}{e_G(X_r/W_d)}$$

where ω is a class on W_d.

It is often necessary to work over a larger field than $\mathbf{C}$ for coefficients of cohomology groups. For example when we consider the case of the equivariant Chern polynomial c_T, a formal variable x is introduced. In this case we replace everywhere the scalars $\mathbf{C}$ by $\mathbf{C}(x)$. This will be implicit in all of the discussion below.

Recall the localization formula:

$$\int_{W_d}\omega = \sum_{0\preceq r\preceq d}\int_X \frac{j_r^*(\omega)}{e_G(X_r/W_d)}.$$

We shall often apply the following version:

$$\int_{W_d}\omega\ e^{\kappa_\zeta} = \sum_{0\preceq r\preceq d}\int_X i_r^*\omega^v\ e^{H_\zeta + \langle H_\zeta, r\rangle\alpha}.$$

Definition 4.1. *Fix an invertible class $\Omega \in H_T^*(X)^{-1}$. A list $P:\ P_d \in H_G^*(W_d)^{-1}$, $d \succ 0$, is a Ω-Euler data if on X,*

$$\Omega\ i_r^*P_d^v = \overline{i_0^*P_r^v}\ i_0^*P_{d-r}^v$$

(called Euler data identity) for all $r \preceq d$, and the $\int_{W_d} P_d\cdot\omega$ are polynomial in α for all $\omega \in R_d$. By convention we set $P_0 = \Omega$.

Example 0. In the last section we have proved, using the *gluing identity*, that the data $Q:\ \ Q_d = \varphi_!(\pi^* b_T(V_d))$ associated with a mixed bundle V and a multiplicative class b_T satisfies the Euler data identity. This indicates that the gluing identity is really the

geometric origin of Euler data. This is what motivates our definition of Euler data. Note that since Q_d is the equivariant push-forward of a class in $H_G^*(M_d(X))$, the polynomial condition on Q is automatic. This condition will be needed when mirror transformation is discussed.

Example 1. Let L be any equivariant line bundle with $c_1(L) \geq 0$. Let $\hat{L}$ be the G-equivariant lift of $c_1(L)$.

$$P_d = \prod_{k=0}^{\langle c_1(L), d\rangle} (\hat{L} - k\alpha)$$

is an Ω-Euler data where $\Omega = c_1(L)$.

Example 2. Let L be any equivariant line bundle with $c_1(L) < 0$. Let $\hat{L}$ be the G-equivariant lift of $c_1(L)$.

$$P_d = \prod_{k=1}^{-\langle c_1(L), d\rangle - 1} (\hat{L} + k\alpha)$$

is an Ω-Euler data where $\Omega = c_1(L)^{-1}$.

Example 3. If P, P' are Ω, Ω'-Euler data respectively, then $P \cdot P'$ is a $\Omega\Omega'$-Euler data as shown in [37].

Example 4. Let L be as in Example 1, and x be a formal variable. Then

$$P_d = \prod_{k=0}^{\langle c_1(L), d\rangle} (x + \hat{L} - k\alpha)$$

is an Ω-Euler data where $\Omega = c_T(L)$ denotes the Chern polynomial.

In each of Examples 1-4 above, the Euler data identity follows immediately from the algebraic identity $\Omega\ j_r^* P_d = \overline{j_0^* P_r}\ j_0^* P_{d-r}$, and Lemma 3.3.

Strictly speaking, in the examples above, we must require that $c_1(L)$ be an invertible class. This requirement can be easily met by twisting L by a trivial line bundle on which T acts by a suitable weight. In the end, we will only be interested in the nonequivariant limit of an Euler data. Thus the choice of twisting is of no consequence at the end. Alternatively, we can consider the Chern polynomial or the total Chern class (which is automatically invertible) instead of the first Chern class.

4.1. An algebraic property

Let $\mathcal{S}$ denotes the set of sequences $B: \quad B_d \in H_G^*(X)^{-1}, \quad d \succeq 0$. By convention, we set $B_0 = \Omega$.

Definition 4.2. *Given any $B \in \mathcal{S}$, define the formal series*

$$HG[B](t) := e^{-H\cdot t/\alpha}\left(\Omega + \sum_{d \succ 0} B_d\ e^{d\cdot t}\right).$$

Note that $e^{H\cdot t/\alpha}HG[B](t)$ takes value in the ring $H_G(X)^{-1}[[K^\vee]]$. (Notations: if R is a ring, hen $\mathcal{R}[[K^\vee]] := \{\sum_{d\in\Lambda} a_d e^{d\cdot t} | a_d \in \mathcal{R}\}$. We use the notations $e^{d\cdot t} = e^{\langle H_t, d\rangle}$ interchangeably.)

Let P be an Euler data, and let B be the list with $B_d := i_0^* P_d^v$. By the localization formula and the Euler data identity, we have

$$\begin{aligned}\int_{W_d} P_d\ e^{\kappa\zeta} &= \sum_{r\preceq d} \int_X i_r^* P_d^v\ e^{H_\zeta + \langle H_\zeta, r\rangle \alpha} \\ &= \sum_{r\preceq d} e^{-d\cdot\tau} \int_X \Omega^{-1} \overline{\left[e^{-H_t/\alpha} i_0^* P_r^v\ e^{r\cdot t}\right]}\ \left[e^{-H_\tau/\alpha} i_0^* P_{d-r}^v\ e^{(d-r)\cdot\tau}\right].\end{aligned}$$

Here $t = \zeta\alpha + \tau$. Note that $\bar{\zeta} = -\zeta$, $\bar{\alpha} = -\alpha$, and all other variables are invariant under the "bar" operation. Now multiply both sides by $e^{d\cdot\tau}$ and sum over $d \in K^\vee$, we get the formula:

$$\sum_d e^{d\cdot\tau} \int_{W_d} P_d\ e^{\kappa\zeta} = \int_X \Omega^{-1}\ \overline{HG[B](\zeta\alpha+\tau)}\ HG[B](\tau). \tag{4.1}$$

By definition, the coefficient of $e^{d\cdot\tau}$ on the right hand side is a power series in ζ with coefficients which are polynomial in α, ie. the series lies in $\mathcal{R}[[e^\tau, \zeta]]$.

Conversely, given $B \in \mathcal{S}$ such that

$$\int_X \Omega^{-1}\ \overline{HG[B](\zeta\alpha+\tau)}\ HG[B](\tau) \in \mathcal{R}[[e^\tau, \zeta]],$$

there exists a unique Euler data $P: \quad P_d$ satisfying (4.1). Namely, P_d is defined by the conditions

$$j_r^* P_d = \Omega^{-1} e_G(X_r/W_d)\ \overline{B_r}\ B_{d-r}.$$

Thus an Euler data P gives rise to a list $B \in \mathcal{S}$ in a canonical way. Abusing the terminology, *we shall call such a B an Euler data.*

5. Linking and Uniqueness

Lemma 5.1. *Let $\omega \in H_T^*(X)^{-1}(\alpha)$. Suppose that*
(a) for each $q \in X^T$, $\omega_q(\alpha) := \omega(\alpha)|_q$ is a Laurent polynomial in α with $deg_\alpha \omega(\alpha) \le -2$;
(b) the power series in ζ: $\int_X \left(\omega(\alpha)e^{H_\zeta} + \omega(-\alpha)e^{H_\zeta + \langle H_\zeta, d\rangle\ \alpha}\right)$ has coefficients which are polynomial in α.
Then $\omega = 0$.

Proof: Suppose $\omega \neq 0$, and we will get a contradiction. By assumption (a), we can write

$$\frac{\omega_q(\alpha)}{e_T(q/X)} = a_q\alpha^{-k} + b_q\alpha^{-k+1} + \cdots$$

which is a *finite* sum, with a_q independent of α and $k \ge 2$. By supposition, not all the a_q are zero. By localization, we get

$$\int_X \left(\omega(\alpha)e^{H_\zeta} + \omega(-\alpha)e^{H_\zeta + \langle H_\zeta, d\rangle\ \alpha}\right)$$
$$= \sum_q \left((a_q\alpha^{-k} + b_q\alpha^{-k+1} + \cdots)e^{H_\zeta(q)} + (a_q(-\alpha)^{-k} + b_q(-\alpha)^{-k+1} + \cdots)e^{H_\zeta(q) + \langle H_\zeta, d\rangle \alpha}\right).$$

By assumption (b), order by order in ζ, this expression is polynomial in α. Since $k \ge 2$, the polar term with α^{-k} must vanish, and so

$$\sum_q a_q e^{H_\zeta(q)}(1 + (-1)^k) = 0.$$

Since not all a_q are zero and the functions $e^{H_\zeta(q)}$ are linearly independent, it follows that k is odd. Now the coefficient of α^{-k+1} becomes

$$\sum_q e^{H_\zeta(q)}(2b_q - a_q\langle H_\zeta, d\rangle) = 0.$$

Again by linear independence of the exponential functions, it follows that $a_q = 0 = b_q$ for all q, which is a contradiction. $\square$

Lemma 5.2. *Suppose A, B are Euler data with $A_r = B_r$ for all $r \prec d$. Suppose that the $(A_d - B_d)|_q$, $q \in X^T$, are Laurent polynomial in α. Suppose also that $deg_\alpha(A_d - B_d) \le -2$. Then $A_d = B_d$.*

Proof: It suffices to show that

$$\omega(\alpha) := A_d - B_d$$

has property (b) of the preceding lemma.

Let A'_d be the coefficient of $e^{d\cdot\tau}$ in the series

$$\int_X \Omega^{-1}\overline{HG[A](\zeta\alpha+\tau)}HG[A](\tau).$$

Likewise for the B'_d. Since A, B are Euler data, A'_d, B'_d are power series in ζ with coefficients which are polynomial in α. Explicitly,

$$A'_d = \sum_{r\preceq d}\int_X \Omega^{-1}e^{H_\zeta}A_r e^{r\cdot\zeta\alpha}\ A_{d-r}$$

and likewise for the B'_d. Using that $A_r = B_r$, $r \prec d$, and that $A_0 = B_0 = \Omega$, we see that $A'_d - B'_d$ is a sum over r with only two surviving terms, corresponding to $r = 0, d$. That is,

$$A'_d - B'_d = \int_X \left(\omega(\alpha)e^{H_\zeta} + \omega(-\alpha)e^{H_\zeta+\langle H_\zeta, d\rangle\ \alpha}\right).$$

Since both A'_d, B'_d have coefficients which are polynomial in α, this shows that the class $\omega(\alpha)$ has property (b) of the preceding lemma. $\square$

Definition 5.3. *Two Euler data A, B are linked if for every balloon pq in X and every $d = \delta[pq] \succ 0$,*

$$(A_d - B_d)|_q$$

is regular at $\alpha = \frac{\lambda}{\delta}$ where λ is the weight on the tangent line $T_q(pq)$.

Suppose A, B both come from Euler data Q, P respectively, ie. $A_d = i_0^* Q_d^v$ and $B_d = i_0^* P_d^v$. Suppose also that

$$j_0^*(P_d)|_q = j_0^*(Q_d)|_q \quad \textit{at } \alpha = \lambda/\delta. \tag{5.1}$$

whenever $d = \delta[pq] \succ 0$ as above. Recall that $\alpha = \lambda/\delta$ is at worst a simple pole of $1/e_G(X_0/W_d)|_q$. It follows that $(A_d - B_d)|_q$ is regular at this value. This shows that the conditions (5.1) guarantee that A, B are linked.

Theorem 5.4. *Suppose A, B are linked Euler data satisfying the following properties: for $d \succ 0$,*

(i) If $q \in X^T$, the only possible poles of $(A_d - B_d)|_q$ are scalar multiples of a weight on T_qX.

(ii) $deg_\alpha(A_d - B_d) \leq -2$.

Then $A = B$.

Proof: We will prove, by induction, the assertion that $A_d = B_d$ for all d. If $d = 0$, there is nothing to prove. Suppose the assertion holds for all $r \prec d$. Set $\omega_q(\alpha) := (A_d - B_d)|_q$ as before. We will show, under assumption (i), that the $\omega_q(\alpha)$ are Laurent polynomial in α. It follows then, from the preceding lemma and assumption (ii), that $A_d = B_d$.

Let $\lambda \in \mathcal{T}^* - 0$. We will show that each $\omega_q(\alpha)$ is regular at $\alpha = \lambda$. Recall the power series in ζ: A'_d, B'_d, with coefficients polynomial in α as in the preceding proof. Thus for any integers $k, l \geq 0$,

$$Res_{\alpha=\lambda}\left((\alpha - \lambda)^k(\alpha + \lambda)^l(A'_d - B'_d)\right) = 0.$$

Also recall that

$$\begin{aligned} A'_d - B'_d &= \int_X \left(\omega(\alpha)e^{H_\zeta} + \omega(-\alpha)e^{H_\zeta + \langle H_\zeta, d\rangle\ \alpha}\right) \\ &= \sum_{q \in X^T} \frac{1}{e_T(q/X)} \left(\omega_q(\alpha)\ e^{H_\zeta(q)} + \omega_q(-\alpha)\ e^{H_\zeta(q) + \langle H_\zeta, d\rangle \alpha}\right). \end{aligned}$$

From the preceding two equations, we get

$$\begin{aligned} 0 = \sum_{q \in X^T} \frac{1}{e_T(q/X)} &\Big(e^{H_\zeta(q)}\ Res_{\alpha=\lambda}(\alpha - \lambda)^k(\alpha + \lambda)^l\omega_q(\alpha) \\ &+ e^{H_\zeta(q) + \langle H_\zeta, d\rangle\lambda} Res_{\alpha=\lambda}(\alpha - \lambda)^k(\alpha + \lambda)^l\omega_q(-\alpha)\Big). \end{aligned} \tag{5.2}$$

If $Res_{\alpha=\lambda}(\alpha - \lambda)^k(\alpha + \lambda)^l\omega_q(-\alpha) = 0$ for all q, then the preceding equation shows that $Res_{\alpha=\lambda}(\alpha-\lambda)^k(\alpha+\lambda)^l\omega_q(\alpha) = 0$ for all q, because the vectors $H_\zeta(q)$ are distinct. Similarly if $Res_{\alpha=\lambda}(\alpha-\lambda)^k(\alpha+\lambda)^l\omega_q(\alpha) = 0$ for all q, then we have $Res_{\alpha=\lambda}(\alpha-\lambda)^k(\alpha+\lambda)^l\omega_q(-\alpha) = 0$ for all q. In either case, we conclude that each $\omega_q(\alpha)$ is regular at $\alpha = \lambda$. So if $\alpha = \lambda$ is a pole of a $\omega_q(\alpha)$, then we necessarily have $Res_{\alpha=\lambda}(\alpha - \lambda)^k(\alpha + \lambda)^l\omega_q(\alpha) \neq 0$ *and* $Res_{\alpha=\lambda}(\alpha - \lambda)^k(\alpha + \lambda)^l\omega_p(-\alpha) \neq 0$ for some p, q and some l, k, such that

$$H_\zeta(q) = H_\zeta(p) + \langle H_\zeta, d\rangle\lambda,$$

to ensure cancellation of the exponential functions in (5.2). Note that since $d \succ 0$ and $\lambda \neq 0$, we have $p \neq q$. By our assumption (i), the pole $\alpha = \lambda$ of $\omega_q(\alpha)$ must be of the form $\lambda = \frac{\lambda'}{\delta} \neq 0$ for some weight λ' on T_qX, and some scalar $\delta \neq 0$. By Lemma 2.4, p, q must

be joined by a balloon, $d = \delta[pq]$, and λ' is the weight on the tangent line $T_q(pq)$. Thus if d is not a multiple of $[pq]$, then we have shown that $\omega_q(\alpha)$ is regular away from $\alpha = 0$.

Now suppose that $d = \delta[pq]$, and consider the only possible pole of $\omega_q(\alpha)$ at $\alpha = \frac{\lambda'}{\delta} \neq 0$, as above. By hypothesis, A, B are linked. But this means that $\omega_q(\alpha)$ is regular at $\alpha = \frac{\lambda'}{\delta} \neq 0$. $\Box$

Remark 5.5. *In our applications later, the situation is better then the conditions (i)-(ii) demand. We will have two Euler data A, B such that A_d, B_d separately, rather than just $A_d - B_d$, will satisfy both conditions (i)-(ii) at the outset. In this situation, to prove that $A = B$, it suffices to prove that they are linked.*

6. Mirror Transformations

Throughout this section, we fix an invertible class Ω on X, and will denote by $\mathcal{A}$ the set of Ω-Euler data.

Definition 6.1. *A map $\mu : \mathcal{A} \to \mathcal{A}$ is called a mirror transformation if it preserves linking. In other words, $\mu(A)$ and A are linked for any $A \in \mathcal{A}$. We call $\mu(A)$ a mirror transform of A.*

We now consider a construction of mirror transformations, as motivated by the classic example of [12]. Consider a transformation $\mu : \mathcal{S} \to \mathcal{S}$, $B \to \tilde{B}$, of the type

$$\tilde{B}_d = B_d + \sum_{r \prec d} a_{d,r} B_r \tag{6.1}$$

where the $a_{d,r} \in H_G^*(X)^{-1}$ are a given set of coefficients. This transformation is obviously invertible, and preserves $B_0 = \Omega$.

Lemma 6.2. *Suppose that $B, \tilde{B}$ are both Euler data. Let $d = \delta[pq] \succ 0$ for some balloon pq in X. Suppose that the coefficients in (6.1) are such that their restrictions $a_{d,r}(q)$, $r \prec d$, to the fixed point q are regular at $\alpha = \lambda/\delta$ where λ is the weight on $T_q(pq)$. Then $(\tilde{B}_d - B_d)|_q$ is regular at $\alpha = \lambda/\delta$.*

Proof: From (6.1), it suffices to show that the functions $B_r|_q$, $0 \prec r \prec d$, are regular at $\alpha = \lambda/\delta$. Suppose the contrary that some $B_r|_q = 0$ has a pole of order $k+1$ there. Since B is a Euler data, we know that

$$B'_r := \sum_{s \preceq r} \int_X \Omega^{-1} e^{H_\zeta} B_s e^{\langle H_\zeta, s \rangle \alpha} \; B_{r-s}$$

is a power series in ζ with coefficients polynomial in α.

By the localization formula,

$$B'_r = \sum_{s \preceq r} \sum_{o \in X^T} \frac{1}{e_T(o/X)} e^{H_\zeta(o) + \langle H_\zeta, s\rangle \alpha} \Omega(o)^{-1} B_s(o) B_{r-s}(o).$$

Now multiply both sides by $(\alpha - \lambda/\delta)^k$ and take residue at $\alpha = \lambda/\delta$. We get

$$0 = \sum_{s \preceq r} \sum_{o \in X^T} \frac{1}{e_T(o/X)} e^{H_\zeta(o) + \langle H_\zeta, s\rangle \lambda/\delta} Res_{\alpha=\lambda/\delta} (\alpha - \lambda/\delta)^k \Omega(o)^{-1} B_s(o) B_{r-s}(o).$$

By assumption, the summand above with $s = 0, o = q$ is nonzero. Observe that this term has an exponential factor $e^{H_\zeta(q)}$. Thus in order to cancel this term, any other term contributing to this cancellation must have an identical exponential factor. This means that

$$H_\zeta(q) = H_\zeta(o) + \langle H_\zeta, s\rangle \lambda/\delta$$

for some s with $s \preceq r$, and some $o \in X^T$. By Lemma 2.4, this implies that $s = \delta[pq]$, contradicting that $s \preceq r \prec d$. $\square$

Definition 6.3. *The transformation (6.1) is said to have the regularity property if for every balloon pq in X and $d = \delta[pq]$, the coefficients are such that their restrictions $a_{d,r}(q)$, $r \prec d$, are regular at $\alpha = \lambda/\delta$ where λ is the weight on $T_q(pq)$.*

Thus the preceding lemma says that transformation (6.1) having the regularity property preserves linking.

Again, motivated by [12] and [28], we consider the following special types of transformations. Given a power series $f \in \mathcal{R}[[K^\vee]]$ with no constant term, we have an invertible transformation $\mu_f : \mathcal{S} \to \mathcal{S}$, $B \mapsto \tilde{B}$, such that

$$e^{f/\alpha}\ HG[B](t) = HG[\tilde{B}](t).$$

In fact, we have

$$\tilde{B}_d = B_d + \sum_{r \prec d} f_{d-r} B_r$$

where $e^{f/\alpha} = \sum_{s \succeq 0} f_s e^{s \cdot t}$, $f_s \in \mathcal{R}[\alpha^{-1}]$. This is clearly a transformation of type (6.1) having the regularity property. (In fact, all the coefficients f_{d-r} are regular away from $\alpha = 0$.)

Given power series $g = (g_1, .., g_m)$, $g_j \in \mathcal{R}[[K^\vee]]$ with no constant term, we have an invertible transformation $\nu_g : \mathcal{S} \to \mathcal{S}$, $B \mapsto \tilde{B}$, such that

$$HG[B](t + g) = HG[\tilde{B}](t).$$

In fact since

$$HG[B](t + g) = e^{-H \cdot t/\alpha} e^{-H \cdot g/\alpha} \sum_{d \succeq 0} B_d \; e^{d \cdot t} e^{d \cdot g},$$

if we write $e^{d \cdot g} = \sum_{s \succeq 0} g_{d,s} e^{s \cdot t}$, $g_{d,s} \in \mathcal{R}$ and $e^{-H \cdot g/\alpha} = \sum_{s \succeq 0} \hat{g}_s e^{s \cdot t}$, $\hat{g}_s \in \mathcal{R}[H/\alpha]$, then

$$\tilde{B}_d = B_d + \sum_{r \prec d} a_{d,r} B_r$$

where the $a_{d,r} \in H_G^*(X)^{-1}$ are quadratic expressions in the $g, \hat{g}$. Thus we obtain another transformation $\mathcal{S} \to \mathcal{S}$ of type (6.1), again having the regularity property.

Theorem 6.4. *The transformations $\mu_f, \nu_g : B \mapsto \tilde{B}$ above each defines a mirror transformation. That is, if B is a Euler data then $\mu_f(B)$ and $\nu_g(B)$ are both Euler data linked to B.*

Proof: Let B be a given Euler data. We have seen that the preceding lemma guarantees that $\mu_f(B)$, $\nu_g(B)$ are linked to B. So it suffices to show that they are Euler data.

First case: set $\tilde{B} = \nu_g(B)$, ie.

$$HG[\tilde{B}](t) = HG[B](t + g(e^t)). \tag{6.2}$$

(Here e^t means the variables $(e^{t_1}, .., e^{t_m})$.) Set $t = \zeta\alpha + \tau$, $q_i = e^{\tau_i}$. On the one hand, we have

$$\int_X \Omega^{-1} \; \overline{HG[B](t)} \; HG[B](\tau) = \sum_{d,m} q^d \; \zeta^m \; B'_{d,m}(\alpha) \tag{6.3}$$

for some $B'_{d,m} \in \mathcal{R}$. Now compare

$$(*) \quad \int_X \Omega^{-1} \; \overline{HG[B](t)} \; HG[B](\tau) = \int_X \Omega^{-1} e^{H_\zeta} \sum \bar{B}_d e^{d \cdot \tau} e^{d \cdot \zeta\alpha} \times \sum B_d e^{d \cdot \tau}$$

$$\begin{aligned}(**) \quad \int_X \Omega^{-1} \; \overline{HG[B](t + g(e^t))} \; HG[B](\tau + g(e^\tau)) = & \int_X \Omega^{-1} e^{H_\zeta} e^{H \cdot (\bar{g}(qe^{\zeta\alpha}) - g(q))/\alpha} \\ & \times \sum \bar{B}_d e^{d \cdot (\tau + g(q))} e^{d \cdot (\bar{g}(qe^{\zeta\alpha}) - g(q)) + d \cdot \zeta\alpha} \\ & \times \sum B_d e^{d \cdot (\tau + g(q))}.\end{aligned}$$

This shows that the series (**) can be obtained from (*) by the replacements $\tau \mapsto \tau + g(q)$, $\zeta \mapsto \zeta + (\bar{g}(qe^{\zeta\alpha}) - g(q))/\alpha$. Thus combining (6.2) and (6.3), we get

$$\int_X \Omega^{-1}\ \overline{HG[\tilde{B}](t)}\ HG[\tilde{B}](\tau) = \sum_{d,m} q^d e^{d\cdot g(q)}\ \left(\zeta + (\bar{g}(qe^{\zeta\alpha}) - g(q))/\alpha\right)^m\ B'_{d,m}(\alpha). \quad (6.4)$$

Now write $g = g_+ + g_-$ with $\bar{g}_\pm = \pm g_\pm$. Obviously for any $g(q) \in \mathcal{R}[[q]]$, $g_+(qe^{\zeta\alpha}) - g_+(q) \in \alpha \cdot \mathcal{R}[[q,\zeta]]$. Since the involution $\omega \mapsto \bar{\omega}$ on $\mathcal{R}$ simply changes the sign of α, the fact that g_- is odd shows that $g_-(q) \in \alpha \cdot \mathcal{R}[[q]]$. Likewise for $g_-(qe^{\zeta\alpha})$. This shows that (6.4) lies in $\mathcal{R}[[q,\zeta]]$. This completes our proof in this case.

Second case: set $\tilde{B} = \mu_f(B)$, ie.

$$HG[\tilde{B}](t) = e^{f/\alpha}\ HG[B](t).$$

Again writing $f \in \mathcal{R}[[e^t]]$ as $f = f_+ + f_-$ with $\bar{f}_\pm = \pm f_\pm$, we get

$$\begin{aligned}\int_X \Omega^{-1}\ \overline{HG[\tilde{B}](t)}\ HG[\tilde{B}](\tau) &= e^{-\bar{f}(e^t)/\alpha}\ e^{f(e^\tau)/\alpha} \int_X \Omega^{-1}\ \overline{HG[B](t)}\ HG[B](\tau) \\ &= e^{-(f_+(qe^{\zeta\alpha}) - f_+(q))/\alpha}\ e^{(f_-(qe^{\zeta\alpha}) + f_-(q))/\alpha} \\ &\times \int_X \Omega^{-1}\ \overline{HG[B](t)}\ HG[B](\tau).\end{aligned}$$

The right hand side lies in $\mathcal{R}[[q,\zeta]]$ as before. $\square$

All mirror transformations we will use later will be of the type μ_f, ν_g as above. Moreover, all Euler data we will encounter will have property (i) of Theorem 5.4. The transformations μ_f, ν_g clearly preserve this property.

Theorem 6.5. *Suppose that A, B have property (i) of Theorem 5.4, and that A, B are linked. Suppose that A is an Euler data with $deg_\alpha A_d \le -2$ for all $d \prec 0$, and that there exists power series $f \in \mathcal{R}[[K^\vee]]$, $g = (g_1, .., g_m)$, $g_j \in \mathcal{R}[[K^\vee]]$, all without constant term, such that*

$$e^{f/\alpha} HG[B](t) = \Omega - \Omega \frac{H \cdot (t+g)}{\alpha} + O(\alpha^{-2}) \quad (6.5)$$

when expanded in powers of α^{-1}. Then

$$HG[A](t+g) = e^{f/\alpha}\ HG[B](t).$$

Proof: By Theorem 6.4, f, g define two mirror transformations μ_f, ν_g, with

$$\begin{aligned} HG[\tilde{B}](t) &= e^{f/\alpha} HG[B](t) \\ HG[\tilde{A}](t) &= HG[A](t+g) \end{aligned} \tag{6.6}$$

where $\tilde{B} = \mu_f(B)$, $\tilde{A} = \nu_g(A)$. Now both $\tilde{B}, \tilde{A}$ have property (i) of Theorem 5.4. (See remark after Theorem 6.4.)

Since $deg_\alpha A_d \leq -2$, $HG[\tilde{A}](t)$ has the same asymtotic form as $HG[\tilde{B}](t)$ in eqn. (6.5) $mod\ O(\alpha^{-2})$. It follows that

$$e^{H\cdot t/\alpha}\ HG[\tilde{A} - \tilde{B}](t) \equiv O(\alpha^{-2}),$$

or equivalently $deg_\alpha(\tilde{A}_d - \tilde{B}_d) \leq -2$. Thus $\tilde{A}, \tilde{B}$ satisfy condition (ii) of Theorem 5.4. Since A is linked to B, it follows that $\tilde{A}$ is linked to $\tilde{B}$. By Theorem 5.4, we conclude that $\tilde{A} = \tilde{B}$. Now our assertion follows from eqns. (6.6). $\Box$

Remark 6.6. *The preceding theorem says that one way to compute A (or Q) is by first finding an explicit Euler data B linked to A, and then relate A and B via mirror transformations.*

7. From stable map moduli to Euler data

Fix an admissible balloon manifold with $c_1(X) \geq 0$. Fix a T-equivariant multiplicative class b_T. Its nonequivariant limit is denoted by b. Fix a T-equivariant bundle of the form $V = V^+ \oplus V^-$, where $V^\pm$ are respectively the convex/concave bundles. As before, we write

$$\Omega = \frac{b_T(V^+)}{b_T(V^-)}.$$

Let V_d be the bundle induced by V on the 0-pointed degree d stable map moduli of X. Throughout this section, we denote

$$\begin{aligned} Q: \quad Q_d &:= \varphi_!(\pi^* b_T(V_d)) \\ K_d &:= \int_{M_{0,0}(d,X)} b(V_d) \\ \Phi &:= \sum K_d\ e^{d\cdot t} \\ A: \quad A_d &:= i_0^* Q_d^v. \end{aligned}$$

Note that all these objects depend on the choice of b_T and V, though the notations do not reflect this.

Theorem 7.1. *(i)* $deg_\alpha A_d \le -2$.
(ii) If for each d the class $b_T(V_d)$ has homogeneous degree the same as the degree of $M_{0,0}(d,X)$, then in the nonequivariant limit we have

$$\int_X e^{-H\cdot t/\alpha} A_d = \alpha^{-3}(2 - d\cdot t)K_d$$
$$\int_X \left(HG[A](t) - e^{-H\cdot t/\alpha}\Omega\right) = \alpha^{-3}(2\Phi - \sum t_i \frac{\partial\Phi}{\partial t_i}).$$

Proof: Earlier we have proved that

$$A_d = i_0^* Q_d^v = ev_! \left(\frac{\rho^* b_T(V_d)}{\alpha(\alpha - c_1(L))}\right),$$

where $L = L_d$ is the line bundle on $\mathcal{M}_{0,1}(d,X)$ whose fiber at a point $(f,C;x)$ is the tangent line at x.

Assertion (i) now follows immediately from this formula. The second equality in assertion (ii) follows from the first equality. By the above formula again,

$$\begin{aligned} I &:= \int_X e^{-H\cdot t/\alpha} A_d \\ &= \int_{M_{0,1}(d,X)} e^{-ev^* H\cdot t/\alpha} \frac{\rho^* b(V_d)}{\alpha(\alpha - c_1(L))} \\ &= \int_{M_{0,0}(d,X)} b(V_d)\ \rho_! \left(\frac{e^{-ev^* H\cdot t/\alpha}}{\alpha(\alpha - c_1(L))}\right). \end{aligned}$$

Now $b(V_d)$ has homogeneous degree the same as the dimension $M_{0,0}(d,X)$. The second factor in the last integrand contributes a scalar factor given by integration over a generic fiber E (which is a $\mathbf{P}^1$) of ρ. So we pick out the degree 1 term in $\frac{e^{-ev^* H\cdot t/\alpha}}{\alpha(\alpha - c_1(L))}$, which is just $\frac{-ev^* H\cdot t}{\alpha^3} + \frac{c_1(L)}{\alpha^3}$. Restricting to the generic fiber E, say over $(f,C) \in M_{0,0}(d,X)$, the evaluation map ev is equal to f, which is a degree d map $E \cong \mathbf{P}^1 \to X$. It follows that

$$\int_E ev^* H = d.$$

Moreover, since $c_1(L)$ restricted to E is just the first Chern class of the tangent bundle to E, it follows that

$$\int_E c_1(L) = 2.$$

So we have

$$I = (-\frac{d \cdot t}{\alpha^3} + \frac{2}{\alpha^3})K_d. \quad \Box$$

Theorem 7.2. *More generally suppose b_T is an equivariant multiplicative class of the form*

$$b_T(V) = x^r + x^{r-1}b_1(V) + \cdots + b_r(V), \quad rk\ V = r$$

where x is a formal variable, b_i is a characteristic class of degree i. Suppose $s := rk\ V_d - dim\ M_{0,0}(d, X) \geq 0$ is independent of $d \succ 0$. Then

$$\frac{1}{s!}\left(\frac{d}{dx}\right)^s |_{x=0} \int_X e^{-H \cdot t/\alpha} A_d = \alpha^{-3} x^{-s}(2 - d \cdot t)K_d$$

$$\frac{1}{s!}\left(\frac{d}{dx}\right)^s |_{x=0} \int_X \left(HG[A](t) - e^{-H \cdot t/\alpha}\Omega\right) = \alpha^{-3} x^{-s}(2\Phi - \sum t_i \frac{\partial \Phi}{\partial t_i}).$$

Proof: The proof is entirely analogous to (ii) above. $\Box$

7.2. Linking theorem for A

Now consider a mixed bundle $V = V^+ \oplus V^-$ on X. Fixed a choice of equivariant multiplicative class b_T. We assume that V has the following property: there exists nontrivial T-equivariant line bundles $L_1^+, .., L_{N^+}^+; L_1^-, .., L_{N^-}^-$ on X with $c_1(L_i^+) \geq 0$ and $c_1(L_j^-) < 0$, such that for any balloon $pq \cong \mathbf{P}^1$ in X we have

$$V^{\pm}|_{pq} = \oplus_{i=1}^{N^{\pm}} L_i^{\pm}|_{pq}.$$

Note that $N^{\pm} = rk\ V^{\pm}$. We also require that

$$b_T(V^+)/b_T(V^-) = \prod_i b_T(L_i^+)/\prod_j b_T(L_j^-).$$

In this case we call the list $(L_1^+, .., L_{N^+}^+; L_1^-, .., L_{N^-}^-)$ the splitting type of V. Note that V is not assumed to split over X. Given such a bundle V and a choice of multiplicative class b_T, we obtain an Euler data Q : $Q_d = \varphi_!(\pi^* b_T(V_d))$ (or A) as before.

Theorem 7.3. *Let $b_T = e_T$ be the equivariant Euler class. Let pq be a balloon, $d = \delta[pq] \succ 0$, and λ be the weight on the tangent line $T_q(pq)$. Then at $\alpha = \lambda/\delta$, we have*

$$j_0^*(Q_d)|_q = \prod_i \prod_{k=0}^{\langle c_1(L_i^+), d\rangle} \left(c_1(L_i^+)|_q - k\lambda/\delta\right) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-), d\rangle - 1} \left(c_1(L_j^-)|_q + k\lambda/\delta\right).$$

In particular Q is linked to

$$P: \quad P_d = \prod_i \prod_{k=0}^{\langle c_1(L_i^+), d\rangle} (\hat{L}_i^+ - k\alpha) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-), d\rangle - 1} (\hat{L}_j^- + k\alpha).$$

Proof: We first consider one positive line bundle L. As in [37], we consider a point $(f, C) \in M_d(X)$ where f is δ-cover from $C = \mathbf{P}^1$ to the balloon $pq \simeq \mathbf{P}^1$. For $\alpha = \lambda/\delta$, this map can be written as

$$f: \ C \to \mathbf{P}^1 \times pq \subset \mathbf{P}^1 \times X$$

where the second map is the inclusion. In terms of coordinates we can write the first map as

$$f: \ [w_0, w_1] \to [w_1, w_0] \times [w_0^\delta, w_1^\delta].$$

Note that the T-action induces standard rotation on $pq \simeq \mathbf{P}^1$ with the weights λ_1, λ_2 and $\lambda = \lambda_1 - \lambda_2$. It is now easy to see that this point (f, C) is fixed by the subgroup of G with $\alpha = \lambda/\delta$. On the other hand as argued in [37], $(\pi_2 \circ f, C)$ is then a smooth fixed point in $M_{0,0}(d, X)$ under the T-action. The restriction $j_0^* Q_d|_p$ with $\alpha = \lambda/\delta$ is equal to the value of $e_T(\mathcal{U}_d)$ at (f, C). This, in turn, is equal to the restriction of $e_T(V_d)$ at $(\pi_2 \circ f, C)$ in $M_{0,0}(d, X)$.

Assume the restriction of L to $pq \simeq \mathbf{P}^1$ is $\mathcal{O}(l)$ with $l = \langle c_1(L), [pq]\rangle$. We compute that the equivariant Euler class restricted to this point $(\pi_2 \circ f, C)$. As in [37], we get

$$e_T(U_d) = \prod_{m=0}^{l\delta} (l\lambda_1 - m\frac{\lambda}{\delta}).$$

Also note that $c_1(L)(p) = l\lambda_1$ and $d = \delta[pq]$, this implies that $Q_d = \varphi_!(\pi^* e_T(V_d))$ is linked to

$$P_d = \prod_{m=0}^{\langle c_1(L), d\rangle} (c_1(\hat{L}) - m\alpha).$$

Similarly for a concave line bundle L, if its restriction to the balloon pq is $\mathcal{O}(-l)$ with $-l = \langle c_1(L), [pq]\rangle$, then

$$e_T(U_d) = \prod_{m=-}^{l\delta-1} (-l\lambda_1 + m\frac{\lambda}{\delta})$$

which implies the formula that in this case Q_d is linked to

$$P_d = \prod_{m=1}^{-\langle c_1(L),d\rangle-1} (c_1(\hat{L}) + m\alpha).$$

The general case is just a product of these cases. $\square$

Similarly we can prove the following formula for the Chern polynomial.

Theorem 7.4. *Let $b_T = c_T$ be the equivariant Chern polynomial. Let pq be a balloon, $d = \delta[pq] \succ 0$, and λ be the weight on the tangent line $T_q(pq)$. Then at $\alpha = \lambda/\delta$, we have*

$$j_0^*(Q_d)|_q = \prod_i \prod_{k=0}^{\langle c_1(L_i^+),d\rangle} \left(x + c_1(L_i^+)|_q - k\lambda/\delta\right) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-),d\rangle-1} \left(x + c_1(L_j^-)|_q + k\lambda/\delta\right).$$

In particular Q is linked to

$$P: \quad P_d = \prod_i \prod_{k=0}^{\langle c_1(L_i^+),d\rangle} (x + \hat{L}_i^+ - k\alpha) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-),d\rangle-1} (x + \hat{L}_j^- + k\alpha).$$

8. Applications

8.1. Toric manifolds

We call a toric manifold X *reflexive* if its defining fan satisfies the following combinatorial condition: the convex hull of the primitive generators of the 1-cones in the fan is a reflexive polytope. It has been shown [4][41] that a pair of polar reflexive polytopes gives rise to a pair of mirror (in the sense of Hodge numbers) Calabi-Yau varieties, by taking anti-canonical hypersurfaces in the corresponding reflexive toric manifolds. It has been conjectured that [5] a similar statement holds for complete intersections in toric manifolds. It is known that [29] a toric manifold X is reflexive iff $c_1(X) \geq 0$. We shall assume that X is reflexive. Recall that for a (convex) toric manifold X, we have

$$e_G(X_0/W_d) = \prod_a \prod_{k=1}^{\langle D_a,d\rangle} (D_a - k\alpha)$$

where each D_a is the T-equivariant first Chern classes of the line bundles corresponding to a T-invariant hypersurfaces in X.

8.2. Chern polynomials for mixed bundles

To proceed, we make two further choices: let b_T be the T-equivariant Chern polynomial c_T, and let $V = V^+ \oplus V^-$ be a mixed bundle with splitting type $(L_1^+, .., L_{N^+}^+; L_1^-, .., L_{N^-}^-)$. Here the L's are T-equivariant line bundles on X with

$$
\begin{aligned}
&c_1(L_i^+) \geq 0, \quad c_1(L_j^-) < 0,\\
&\Omega := c_T(V^+)/c_T(V^-) = \prod_i (x + c_1(L_i^+))/\prod_j (x + c_1(L_j^-))\\
&\sum_i c_1(L_i^+) - \sum_j c_1(L_j^-) = c_1(X).
\end{aligned}
$$

From this, we get an Ω-Euler data Q : $Q_d = \varphi_!(\pi^* c_T(V_d))$ as before. By the Linking Theorem, Q is linked to the Euler data

$$
P: \quad P_d = \prod_i \prod_{k=0}^{\langle c_1(L_i^+),d\rangle} (x + \hat{L}_i^+ - k\alpha) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-),d\rangle - 1} (x + \hat{L}_j^- + k\alpha).
$$

As before, we set

$$
B: \quad B_d = i_0^* P_d^v, \qquad A: \quad A_d = i_0^* Q_d^v.
$$

We consider three separate cases. We will be using the elementary formula

$$
\prod_{k=1}^{M} (\frac{\omega}{\alpha} - k) \equiv (-1)^M M! (1 - \frac{\omega}{\alpha} \sum_{k=1}^{M} \frac{1}{k}) \tag{8.1}
$$

where "$\equiv$" here means equal *mod* $O(\alpha^{-2})$, to compute the leading terms of

$$
\begin{aligned}
B_d &= \prod_i \prod_{k=0}^{\langle c_1(L_i^+),d\rangle} (x + c_1(L_i^+) - k\alpha) \times \prod_j \prod_{k=1}^{-\langle c_1(L_i^-),d\rangle - 1} (x + c_1(L_i^-) + k\alpha)\\
&\times \frac{1}{\prod_a \prod_{k=1}^{\langle D_a,d\rangle} (D_a - k\alpha)}\\
&= \Omega \; c_T(V^-) \; \alpha^{-N^-} \prod_i \prod_{k=1}^{\langle c_1(L_i^+),d\rangle} (\frac{x + c_1(L_i^+)}{\alpha} - k) \times \prod_j \prod_{k=1}^{-\langle c_1(L_i^-),d\rangle - 1} (\frac{x + c_1(L_i^-)}{\alpha} + k)\\
&\times \frac{1}{\prod_a \prod_{k=1}^{\langle D_a,d\rangle} (\frac{D_a}{\alpha} - k)}.
\end{aligned} \tag{8.2}
$$

First suppose that $rk\ V^- = N^- \geq 2$. In this case we have

$$deg_\alpha B_d = -rk\ V^- \leq -2$$

and hence

$$HG[B](t) \equiv \Omega - \Omega \frac{H \cdot t}{\alpha}.$$

By Theorem 6.5, we conclude that $A = B$ and $Q = P$. This completes the computation of A and Q in this case.

Now consider the case $rk\ V^- = N^- = 1$, hence V^- is a line bundle. In this case we have

$$\begin{aligned} B_d &\equiv \alpha^{-1}\ \Omega\ (x + c_1(V^-))\ (-1)^{\langle c_1(V^-), d\rangle} (-\langle c_1(V^-), d\rangle - 1)! \frac{\prod_i \langle c_1(L_i^+), d\rangle!}{\prod_a \langle D_a, d\rangle!} \\ &=: \alpha^{-1}\ \Omega\ (\sum_i H_i \phi_{d,i} + \psi_d) \end{aligned}$$

where the $\phi_{d,i} \in \mathbf{Q}$, $\psi_d \in \mathbf{Q}[\mathcal{T}^*, x]$, are determined uniquely by the writing $c_1(V^-) \in H_T^2(X)$ in the last equality, according to the decomposition $H_T^2(X) = \oplus_{i=1}^m \mathbf{Q} H_i \oplus \mathcal{T}^*$. Hence we get

$$e^{-H \cdot t/\alpha}\ B_d \equiv \Omega\ (\alpha^{-1} H \cdot \phi_d + \alpha^{-1} \psi_d).$$

Summing over $d \in K^\vee$, we get

$$\begin{aligned} HG[B](t) &\equiv \Omega(1 - \alpha^{-1} H \cdot (t + F) + \alpha^{-1} G) \\ F &:= -\sum \phi_d\ e^{d \cdot t} \\ G &:= \sum \psi_d\ e^{d \cdot t}. \end{aligned}$$

From this we get

$$e^{-G/\alpha}\ HG[B](t) \equiv \Omega - \Omega \frac{H \cdot (t + F)}{\alpha}$$

By Theorem 6.5, we conclude that

$$e^{-G/\alpha} HG[B](t) = HG[A](t + F). \tag{8.3}$$

This completes our computation of A and Q in this case.

Recall that

$$\begin{aligned} dim\ M_{0,0}(d, X) &= \langle c_1(X), d\rangle + n - 3 \\ rk\ V_d &= \sum_i \langle c_1(L_i^+), d\rangle - \sum_j \langle c_1(L_j^-), d\rangle + N^+ - N_- \\ &= \langle c_1(X), d\rangle + rk\ V^+ - rk\ V^-. \end{aligned}$$

To applied Theorem 7.2, we assume that $rk\ V^+ - rk\ V^- \geq n-3$, and we determine all K_d immediately. Explicitly (in the nonequivariant limit $\mathcal{T}^* \to 0$):

$$\frac{1}{s!}\left(\frac{d}{dx}\right)^s |_{x=0} \int_X \left(e^{-G/\alpha} HG[B](t) - e^{-H\cdot\tilde{t}/\alpha}\Omega\right) = \alpha^{-3}x^{-s}(2\Phi(\tilde{t}) - \sum_i \tilde{t}_i \frac{\partial\Phi(\tilde{t})}{\partial\tilde{t}_i}). \quad (8.4)$$

where $s := rk\ V^+ - rk\ V^- - (n-3)$, $\tilde{t} := t + F(t)$. Note that this same formula applies also when $rk\ V^- \geq 2$, whereby we put $G, F = 0$.

We now consider the case when V is purely convex: $N^- = 0$.

8.3. Convex bundle

We will denote the L_i^+ simply by L_i. Using formulas (8.1) and (8.2), we get

$$\begin{aligned} B_d &\equiv \Omega\frac{\prod_i \langle c_1(L_i), d\rangle!}{\prod_a \langle D_a, d\rangle!}(1 + \alpha^{-1}\sum_a D_a \sum_{k=1}^{\langle D_a, d\rangle} \frac{1}{k} - \alpha^{-1}\sum_i (x + c_1(L_i)) \sum_{k=1}^{\langle c_1(L_i), d\rangle} \frac{1}{k}) \\ &=: \Omega\lambda_d + \alpha^{-1}\sum_i H_i\phi_{d,i} + \alpha^{-1}\psi_d \end{aligned}$$

Here the $\lambda_d, \phi_{d,i} \in \mathbf{Q}$, $\psi_d \in \mathbf{Q}[\mathcal{T}^*, x]$ are determined uniquely by the writing each $D_a, c_1(L_i) \in H^2_T(X)$ in the last equality, according to the decomposition $H^2_T(X) = \oplus_{i=1}^m \mathbf{Q}H_i \oplus \mathcal{T}^*$. Since $e^{-H\cdot t/\alpha} \equiv 1 - \alpha^{-1}H\cdot t$, we get

$$e^{-H\cdot t/\alpha}\ B_d \equiv \Omega(\lambda_d - \alpha^{-1}H\cdot(\lambda_d t - \phi_d) + \alpha^{-1}\psi_d).$$

Summing over $d \in K^\vee$, we get

$$\begin{aligned} HG[B](t) &\equiv \Omega\left(F_0 - \alpha^{-1}H\cdot(F_0 t + F) + \alpha^{-1}G\right) \\ F_0 &:= 1 + \sum \lambda_d\ e^{d\cdot t} \\ F &:= -\sum \phi_d\ e^{d\cdot t} \\ G &:= \sum \psi_d\ e^{d\cdot t}. \end{aligned}$$

Put $f := \alpha\ log\ F_0 - \frac{G}{F_0}$. Then we get

$$e^{f/\alpha}\ HG[B](t) \equiv \Omega - \Omega\frac{H\cdot(t + \frac{F}{F_0})}{\alpha}$$

By Theorem 6.5, we conclude that

$$e^{f/\alpha}\ HG[B](t) = HG[A](t + \frac{F}{F_0}). \quad (8.5)$$

This completes our computation of A and Q in this case.

Again to apply Theorem 7.2, we assume that $rk\ V \geq n-3$, and determine all K_d immediately. Explicitly:

$$\frac{1}{s!}\left(\frac{d}{dx}\right)^s |_{x=0} \int_X \left(e^{f/\alpha}\ HG[B](t) - e^{-H\cdot\tilde{t}/\alpha}\Omega\right) = \alpha^{-3}x^{-s}(2\Phi(\tilde{t}) - \sum_i \tilde{t}_i \frac{\partial\Phi(\tilde{t})}{\partial\tilde{t}_i}). \quad (8.6)$$

where $s := rk\ V - (n-3)$, $\tilde{t} := t + \frac{F(t)}{F_0(t)}$.

Let us now specialize to the case $rk\ V = n-3$ (ie. $s=0$), and $V = \oplus_i L_i$. We can then set $x=0$, so that $b_T = c_T$ becomes the equivariant Euler class e_T, and the K_d is just the intersection numbers for $e(V_d)$. Then the formula (8.6) yields the general formula derived in [28] and in [27], on the basis of the conjectural mirror correspondence. Note that

$$F_0 = \sum \frac{\prod_i \langle c_1(L_i), d\rangle!}{\prod_a \langle D_a, d\rangle!} e^{d\cdot t}$$

is an example of a hypergeometric function [20]. It has been proved in [29] that F_0 is the unique holomorphic period of Calabi-Yau hypersurfaces near the so-called large radius limit. For the purpose of comparison, we should mention that the definition of Φ here differs from the prepotential in [28][27] by a degree three polynomial in $\tilde{t}$, and the definition of the hypergeometric series $HG[B](t)$ here differs from that denoted by $w_0(x,\rho)$ in [28][27] by an irrelevant overall constant factor.

Precursors to the above general formula have been many examples [27][6][13][8]. We now specialize to a few numerical examples which have been frequently studied by both physicists and mathematicians alike.

8.4. A complete intersection in $\mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2$

The complete intersection of degrees $(1,3,0)$, $(1,0,3)$ in this 5-dimensional toric manifold X has been studied in [27] using mirror symmetry, and in [30] computing some of the intersection numbers K_d for the Euler class $b=e$ in terms of modular forms.

From our point of view, that complete intersection correspond to the following choice of convex bundle:

$$V = \mathcal{O}_1(1)\otimes\mathcal{O}_2(3)\oplus\mathcal{O}_1(1)\otimes\mathcal{O}_3(3)$$

where $\mathcal{O}_i(l)$ denotes the pullback of $\mathcal{O}(l)$ from the ith factor. The Kähler cone of X is abviously generated by the hyperplanes H_1, H_2, H_3 from the three factors of X, and hence $K^\vee$ can be identified with the set of $d = (d_1, d_2, d_3) \in \mathbf{Z}^3_{\geq 0}$. We consider intersection

numbers K_d for the Euler class $b = e$ as before. Thus we set $\Omega = e(V) = (H_1+3H_2)(H_1+3H_3)$. The Euler data P we need in eqn. (8.6) is given by

$$P_d = \prod_{k=0}^{d_1+3d_2} (\kappa_1 + 3\kappa_2 - k\alpha) \times \prod_{k=0}^{d_1+3d_3} (\kappa_1 + 3\kappa_3 - k\alpha)$$
$$j_0^*(P_d) = \prod_{k=0}^{d_1+3d_2} (H_1 + 3H_2 - k\alpha) \times \prod_{k=0}^{d_1+3d_3} (H_1 + 3H_3 - k\alpha).$$

The linear sigma model is $W_d = N_d(\mathbf{P}(n)) = N_{d_1,1} \times N_{d_2,2} \times N_{d_3,2}$. The equivariant Euler class, after taking nonequivariant limit with respect to the T action, is given by

$$e_G(X_0/W_d) = \prod_{m=1}^{d_1} (H_1 - m\alpha)^2 \prod_{m=1}^{d_2} (H_2 - m\alpha)^3 \prod_{m=1}^{d_3} (H_3 - m\alpha)^3.$$

Now we can easily write down the hypregeometric series and all the K_d can be computed by our formula (8.6) at once using the obvious intersection form on X, given by the relations:

$$\int_X H_1 H_2^2 H_3^2 = 1, \quad H_1^2 = 1, \; H_2^3 = 1, \; H_3^3 = 1.$$

Once we have the hypergeometric series, the corresponding Picard-Fuchs equation can be easily written down as given in [27].

8.5. $V = \mathcal{O}_1(-2) \otimes \mathcal{O}_2(-2)$ *on* $\mathbf{P}^1 \times \mathbf{P}^1$

Here we denote by $\mathcal{O}_i(l)$ the pullback of $\mathcal{O}(l)$ from the ith factor of $X = \mathbf{P}^1 \times \mathbf{P}^1$. Our bundle V has $rk\ V^+ - rk\ V^- = n - 3 = -1$. Thus we can apply our formula (8.4) with $x = 0$. We put $\Omega = \frac{1}{H_1 H_2}$. The Euler data P in eqn. (8.4) that compute the K_d is now given by:

$$P_d = \prod_{k=1}^{2d_1-1} (-2\kappa_1 + k\alpha) \times \prod_{k=1}^{2d_2-1} (-2\kappa_2 + k\alpha).$$

The corresponding equivariant Euler class, after taking the nonequivariant limit with respext to the T-action is

$$e_G(X_0/W_d) = \prod_{m=1}^{d_1} (H_1 - m\alpha)^2 \prod_{m=1}^{d_2} (H_2 - m\alpha)^2.$$

Again one can immediately write down the hypergeometric series as well as the corresponding Picard-Fuchs equation by using our mirror principle.

9. Generalizations and Concluding Remarks

9.1. A weighted projective space

Consider the following example: the concave bundle $V = O(-6)$ over $\mathbf{P}_{3,2,1}$, $\Omega = \frac{1}{6H}$. This example will be studied in our subsequent paper by using resolution of singularities. This is an example of "local mirror symmetry" studied in physics [33]. The mirror formula there can derived as a special case of our general result. In fact, the Euler data which computes the K_d in this case is determined by

$$j_0^* P_d = \prod_{m=1}^{6d-1} (-6H + m\alpha).$$

The corresponding equivariant Euler class, after taking nonequivariant limit with respect to the T action, is:

$$e_G(X_0/W_d) = \prod_{m=1}^{d} (H - m\alpha) \prod_{m=1}^{2d} (2H - m\alpha) \prod_{m=1}^{3d} (3H - m\alpha).$$

The corresponding hypergeometric series and Picard-Fuchs equation can be immediately written down. It turns out that the hypergeometric series gives the periods of a meromorphic 1-form for a family of elliptic curves [33].

9.2. General projective balloon manifolds

Let X be a projective manifold embedded in $\mathbf{P}(n)$, with a system of homogeneous polynomial defining equations $P(z^1, \cdots, z^n) = 0$, where $z^j = (z_1^j, \cdots, z_{n_j}^j)$. For each P, by taking the coefficients of each monomial $w_0^a w_1^b$ in $P(f^1, \cdots, f^n) = 0$, where $f^j = [f_1^j(w_0, w_1), \cdots, f_{n_j}^j(w_0, w_1)]$ for $j = 1, \cdots, k$ is the tuple of polynomials that define the coordinates of $N_d(\mathbf{P}(n))$, we get several equations of the same degree as P. These equations together define a projective variety, which we denote by $N_d(X)$, in $N_d(\mathbf{P}(n))$.

As discussed earlier, we see that the S^1 fixed point components in $N_d(X)$ are given by the X_r's which are copies of X. We do not know whether the localization formula holds on $N_d(X)$. The localization formula holds if the fixed point components embedded into W_d as local complete intersection subvarieties. It is likely that this is the case for any convex projective manifold. If this is true, then we can take $N_d(X)$ to be the linear sigma model W_d for X. Then our mirror principle may apply readily to compute multiplicative characteristic numbers on $M_{0,0}(d, X)$ in terms of the hypergeometric series.

9.3. A General Mirror Formula

Many of our results so far are proved for projective manifolds without T-action. Here we first discuss a formula for computing the numbers

$$K_d = \int_{M_{0,0}(d,X)} b(V_d)$$

for a general convex projective n-fold X *without T-action*. For simplicity, let's focus on the case when the multiplicative class b is the Chern polynomial c, and V is a direct sum of line bundles on X. There is a similar formulation in the general case. We fix a projective embedding $X \to \mathbf{P}(n)$, as before. Note that the map $\varphi : M_d(X) \to N_d(\mathbf{P}(n))$ is now only S^1-equivariant. Recall that the subvariety $W_d := \varphi(M_d(X)) \subset N_d(\mathbf{P}(n))$ contains as S^1 fixed point components copies of X: X_r, $0 \preceq r \preceq d$. We assume that the localization formula holds on it.

We denote by $e_{S^1}(X_0/W_d)$ the equivariant Euler class of the normal bundle of X_0 in W_d. Let

$$V = V^+ \oplus V^-, \quad V^+ := \oplus L_i^+, \quad V^- := \oplus L_j^-$$

satisfying $c_1(V^+) - c_1(V^-) = c_1(X)$ and $rk(V^+) - rk(V^-) - (n-3) \geq 0$, where the $L_i^\pm$ are respectively convex/concave line bundles on X. Let

$$\Omega = B_0 := c(V^+)/c(V^-) = \prod_i (x + c_1(L_i^+))/\prod_j (x + c_1(L_j^-))$$

$$B_d := \frac{1}{e_{S^1}(X_0/W_d)} \times \prod_i \prod_{k=0}^{\langle c_1(L_i^+),d\rangle} (x + c_1(L_i^+) - k\alpha) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-),d\rangle - 1} (x + c_1(L_j^-) + k\alpha).$$

$$HG[B](t) := \sum B_d e^{d\cdot t}$$

$$\Phi(t) := \sum K_d e^{d\cdot t}.$$

Conjecture 9.1. *There exist unique power series $G(t), F(t)$ such that the following formula holds:*

$$\frac{1}{s!}\left(\frac{d}{dx}\right)^s |_{x=0} \int_X \left(e^{-G/\alpha} HG[B](t) - e^{-H\cdot\tilde{t}/\alpha}\Omega\right) = \alpha^{-3} x^{-s} (2\Phi(\tilde{t}) - \sum_i \tilde{t}_i \frac{\partial \Phi(\tilde{t})}{\partial \tilde{t}_i}).$$

where $s := rk\ V^+ - rk\ V^- - (n-3)$, $\tilde{t} := t + F(t)$. Moreover G, F are determined by the condition that the integrand on the left hand side is of order $O(\alpha^{-2})$.

9.4. Formulas without T-action

One of our key ingredient, the functorial localization formula plays an important role in relating the data on $M_d(X)$ and those on W_d. It turns out that similar formula holds in K-theory. It holds even when X has no group action. This indicates that our method may be extended to compute K-theory multiplicative type characteristic classes on $M_d(X)$ (and ultimately on $M_{0,0}(d, X)$), in terms of certain q-hypergeometric series, even for projective manifolds without group action.

We now write down the relevant localization formulas for convex X without torus action, both in cohomology and in K-theory. The notations and proofs are basically the same as before. Given a manifold X, let's assume that there is a linear sigma model W_d.

Lemma 9.2. *For any equivariant cohomology class ω on $M_d(X)$, the following equality holds on X_r for any $0 \preceq r \preceq d$:*

$$\frac{j_r^* \varphi_!(\omega)}{e_{S^1}(X_r/W_d)} = e_! [\frac{i_r^* \omega}{e_{S^1}(F_r/M_d(X))}].$$

Here $e_{S^1}(\cdot)$ denotes the S^1-equivariant Euler class. As in the cases we have studied earlier, the left hand side of the above formula indicates that when $V = L$ is a line bundle, we should compare the Euler data $Q_d = \varphi_! \pi^* e(V_d)$, to the Euler data given by

$$P_d = \prod_{m=0}^{\langle c_1(L), d \rangle} (c_1(L) - m\alpha).$$

What is left is to develope uniqueness and mirror transformations, which we are unable to achieve at this moment, though they can be easily axiomized.

Now let us look at K-theory formula, which can be proved by using equivariant localization in K-theory. First following the same idea, we get the explicit formula as follows: given any equivariant element V in $K(W_d)$, we have

$$V = \sum_r j_{r!} \frac{j_r^* V}{E_G(X_r/W_d)}$$

where $E_G(X_r/W_d)$ is the equivariant Euler class of the normal bundle of X_r in $N_d(X)$. Here the push-forward and pull-backs by j_r denote the corresponding operations in K-theory. By taking $V = 1$, we get

$$e_![\frac{1}{E_G(F_r/M_d(X))}] = \frac{j_r^*\varphi_!(1)}{E_G(X_r/W_d)}.$$

Second, we have the following lemma; If $\varphi_!(1) = 1$, which is the case if $X = \mathbf{P}^n$, this formula gives explicit formulas for some K-theoretic characteristic numbers of the moduli spaces.

Lemma 9.3. *Given any equivariant element V in $K_G(M_d(X))$, then we have formula*

$$e_![\frac{i_r^*V}{E_G(F_r/M_d(X))}] = \frac{j_r^*(\varphi_!V)}{E_G(X_r/W_d)}$$

where $E_G(\cdot)$ denotes the equivariant Euler class of the corresponding normal bundle in K-group.

In particular we have explicit expressions from the decomposition of the normal bundles:

$$E_G(F_r/M_d(X)) = (1 - e^{\alpha})(1 - e^{-\alpha})(1 - e^{\alpha}L_r)(1 - e^{-\alpha}L_{d-r})$$

and similarly

$$E_G(F_0/M_d(X)) = (1 - e^{\alpha})(1 - e^{\alpha}L_d),\ E_G(F_d/M_d(X)) = (1 - e^{-\alpha})(1 - e^{-\alpha}L_d).$$

For a toric manifold X, we also have the explicit class in K-group,

$$E_G(X_0/W_d) = \prod_a \prod_{m=1}^{\langle D_a, d\rangle} (1 - e^{m\alpha}[D_a])$$

where $[D_a]$ are the equivariant line bundle corresponding to the T divisors D_a.

If V is a multipicative type K-theory characteristic class, then we can develope a similar theory of Euler data and uniqueness. These result can also be extended to the nonconvex case without a hitch.

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